SWITCHING, MEAN-SEEKING, AND RELATIVE RISK

WITH TWO OR MORE RISKY ASSETS

1. Introduction

Ever since the seminal work of Arrow (1965) and Pratt (1964), researchers have recognized the importance of understanding the economic behavior implied by various features of von Neumann-Morgenstern utility functions. In particular, in a positive context it is useful to know if certain utility function features are necessary to imply behavior which may be empirically plausible for most economic actors, in which case such features may be assumed in empirical work; in a normative context, a decision-maker wishing to describe his preferences to an agent acting on his behalf may choose to eliminate from consideration any class of utility functions giving undesired behavior in certain benchmark situations.

In the absence of restrictions on the joint distribution of asset returns, the literature contains almost no results for \( n \)-asset portfolios relating the form of the von Neumann-Morgenstern utility function to the comparative statics of portfolios. One exception is Mitchell and Douglas (1997), who show that the results of Meyer and Ormiston (1994) for two risky assets can be extended to the case of \( n \) assets. Those results are that if the conditional distribution of one asset's return, for all possible realizations of the other assets' returns, undergoes a first-order stochastically dominating shift [or, a mean-preserving contraction], then the optimal portfolio share in the affected asset always goes up or does not change if and only if the utility function \( U \) is such that \( WU'(W) \) is nondecreasing [or, concave].

The other exception is Cass and Stiglitz (1972), who show that the certainty equivalent rate of
return is an increasing, constant, or decreasing function of initial wealth as Arrow-Pratt relative risk
aversion is decreasing, constant, or increasing. In addition, Cass and Stiglitz (1970) identify the class of
utility functions for which, with one riskfree and \((n-1)\) risky assets, a rise in initial wealth leaves the
optimal risky asset proportions unchanged, so that in effect there is a single composite risky asset, and
all results for one riskfree and one risky asset apply.\(^1\)

Part of the reason for the dearth of \(n\)-asset results in the absence of distributional restrictions is
to be found in Hart (1975). He shows that with a riskfree asset, unless the utility function has the
separation property under which \((n-1)\) risky assets are treated as a single composite asset, no utility
function can exhibit a "regular wealth effect comparative statics property." Roughly speaking, this result
precludes any theorem stating that utility functions with some given property (other than the separation
property) always give a comparative static result of the form that \(dz/dW_0\) has a particular sign, where
\(W_0\) is initial wealth and \(z\) is some admissible function of the shares or absolute amounts held in the
assets. However, the scope of Hart's theorem is limited by an important caveat (appearing in his
footnote on page 618): the function defining \(z\) cannot contain parameters of the joint distribution
function. Thus, for example, Hart does not preclude comparative static results for the sign of
\(d(E\tilde{W}/W_0)/dW_0\) where \(\tilde{W}\) is final wealth and \(E\tilde{W}/W_0\) is return per dollar invested. In fact, the present
paper will present a necessary and sufficient condition on the utility function for us to have
\(d(E\tilde{W}/W_0)/dW_0 \leq 0\) for all joint distributions. Such a utility function may be referred to as increasingly
mean-seeking.

A related issue is similar to one arising in Bell (1988). Bell finds the set of utility functions
which, as initial wealth rises, switch preference no more than once between two additive risks of the
form $W_0 + \tilde{x}$ and $W_0 + \tilde{y}$; he also finds the subset of these utility functions which only switch to the higher-mean risk. It is natural in our context to ask a related pair of questions: what utility functions switch only once, as initial wealth rises, between risks of the form $W_0 \tilde{r}$ and $W_0 \tilde{s}$; and, of those, which switch only to the higher-mean risk? In our portfolio context $\tilde{r}$ and $\tilde{s}$ can be interpreted as returns per dollar invested in two alternative portfolios, so $W_0 \tilde{r}$ and $W_0 \tilde{s}$ are total returns of the two portfolios. It is intuitively appealing to think that utility might be such that, if the preference between these two fixed portfolios switches from one to the other as $W_0$ rises, further increases in $W_0$ would not cause a reverse switch back to the originally preferred portfolio. It is also intuitively appealing to think that utility might be such that higher initial wealth never leads to a switch to a portfolio with lower mean return per dollar. The present paper will identify the sets of utility functions exhibiting these properties, and show how they relate to the set of utility functions giving the above-mentioned comparative static result for $n$-asset portfolios.

Bell (1988, 1995) shows a further property of the utility functional form that only permits switches to the higher-mean additive risk: it is amenable to expressing expected utility as a function of mean final wealth and the absolute risk of final wealth, for suitable definition of the latter. Analogously, we will show that the utility functions that only switch to the higher-mean multiplicative risk are the very ones shown by Sarin and Weber (1993) to be amenable to expressing expected utility as a function of mean final wealth and the relative risk of final wealth, for suitable definition of the latter. Moreover, we show that this class of utility functions, to the exclusion of any others, give the previously mentioned property in $n$-asset portfolios that $d(E\tilde{W}/W_0)/dW_0 \geq 0$ always; in addition, this class has the further property that a rise in initial wealth increases (or leaves unchanged) the relative risk of the optimal $n$-
asset portfolio. We refer to the former property as that of being increasingly mean-seeking, and to the latter property as broadly decreasing relative risk aversion.

Thus this paper ties together two strands of the literature--that on switching of preference between two alternatives as initial wealth increases, and that on finding comparative static properties of \( n \)-asset portfolios--while extending both of them. Section 2 identifies the class of utility functions that never switch (Theorem 1), or that switch only once (Theorem 2), between \( W_0 \tilde{r} \) and \( W_0 \tilde{s} \), and identifies in Theorem 3 the subset of the latter which only switch to the risk with higher mean as the utility functions which are the sum of linear and power (or log) terms. Section 3 highlights the observation of Sarin and Weber (1993) that the linear-plus-power and linear-plus-log utilities permit expected utility to be expressed as a well-behaved function of mean final wealth and the relative risk of final wealth, for an appropriate definition of relative risk, and shows (in Theorem 4) that preference switches only to gambles with higher relative risk. Then Section 4 shows (in Theorem 5) that the linear-plus-power and linear-plus-log utilities, and no others, always give the comparative static property that \( d(E\tilde{W}/W_0)/dW_0 \leq 0 \) in \( n \)-asset portfolios; it also shows in Theorem 6 that these utility functions always give a rise or no change in the relative risk of the optimal portfolio when \( W_0 \) rises. Section 5 concludes.

2. Zero-Switch and One-Switch Utility Functions

Bell (1988) identified the set of utility functions which are zero-switch, or one-switch, in the sense that preference between \( W_0 + \tilde{x} \) and \( W_0 + \tilde{y} \) (with additive risks \( \tilde{x} \) and \( \tilde{y} \) and known initial wealth \( W_0 \) ) never switches, or switches no more than once, as \( W_0 \) rises. Here we consider the
corresponding properties in the context of risks which are multiplicative rather than additive: \( W_0 \tilde{r} \) and \( W_0 \tilde{s} \). The random variables \( \tilde{r} \) and \( \tilde{s} \) can be viewed as the per-dollar returns on alternative portfolios, so \( W_0 \tilde{r} \) or \( W_0 \tilde{s} \) is the total portfolio return.

**Definition:** A utility function \( U \) is said to be *multiplicatively zero-switch* (or, *multiplicatively one-switch*) if for all non-negative random variables \( \tilde{r} \) and \( \tilde{s} \), \( EU(W_0 \tilde{r}) \) - \( EU(W_0 \tilde{s}) \) never switches sign (or, switches sign no more than once) as \( W_0 \) rises.

Theorems 1 and 2 identify the classes of utility functions which are multiplicatively zero-switch or one-switch. The proofs of these theorems rely heavily on Bell (1988).

**Theorem 1:** A utility function is multiplicatively zero-switch if and only if it has either the form \( b \ln W + c \) or the form \( aW^b + c \).

**Proof:** To be multiplicatively zero-switch, a utility function \( U \) must never switch preference between \( W_0 \tilde{r} \) and \( W_0 \tilde{s} \) as \( W_0 \) varies. The choice between \( W_0 \tilde{r} \) and \( W_0 \tilde{s} \) is the question of whether \( EU(W_0 \tilde{r}) < \) or \( > EU(W_0 \tilde{s}) \) --that is, whether \( EU(exp(ln W_0 \tilde{r})) < \) or \( > EU(exp(ln W_0 \tilde{s})) \), or equivalently \( EU(exp(ln W_0 + ln \tilde{r})) < \) or \( > EU(exp(ln W_0 + ln \tilde{s})) \), or hence \( EV(ln W_0 + ln \tilde{r}) < \) or \( > EV(ln W_0 + ln \tilde{s}) \) where \( V \equiv U \exp \). But zero-switch behavior of \( V \) between \( ln W_0 + ln \tilde{r} \) and \( ln W_0 + ln \tilde{s} \) as \( W_0 \) varies is the same as that while \( ln W_0 \) varies. Thus there is a correspondence between multiplicatively zero-switch utilities \( U \) and additively zero-switch utilities \( V \), with \( V \equiv U \exp \) or equivalently \( U \equiv V \ln \). By Bell (1988, Proposition 1) the additively zero-switch utilities \( V \) are of the form \( V(Z) = bZ + c \) and \( V(Z) = ae^{bZ} + c \). Letting \( Z \equiv ln W \), we have \( U(W) = b \ln W + c \) and \( U(W) = aW^{b} + c \). QED.

**Corollary 1.1:** The only distinct multiplicatively zero-switch utilities \( U \) with \( U' > 0 \) and
Note that here for the purpose of parsimonious parameterization, use has been made of the fact that increasing linear transformations of a utility function do not lead to a distinct utility function. The utilities identified in Corollary 1.1 are the complete set of constant relative risk aversion (CRRA) utility functions, which of course have decreasing absolute risk aversion (DARA) as well. We now turn to one-switch utilities.

**Theorem 2:** A utility function is multiplicatively one-switch if and only if it has one of the following forms: (i) \( a \ln W + cW^b + k \); (ii) \( aW^c \ln W + bW^c + k \); (iii) \( a(\ln W)^2 + b \ln W + k \); (iv) \( aW^b + cW^d + k \).

**Proof:** By the reasoning in the proof of Theorem 1, the set of multiplicatively one-switch utilities \( U \) corresponds to the set of additively one-switch utilities \( V \) according to \( U / VBln \). By Bell (1988, Proposition 2), the latter set consists of (i) \( aZ + ce^{bZ} + k \); (ii) \( aZe^{cZ} + be^{cZ} + k \); (iii) \( aZ^2 + bZ + k \); and (iv) \( ae^{bZ} + ce^{dZ} + k \). Using \( Z / \ln W \), we obtain the functions \( U \) given in the theorem. QED.

Note that in each of cases (i)-(iv) given in this theorem, the unique switch point can be solved for explicitly as the value of \( W_0 \) which solves \( EU(W_0 \tilde{r}) = EU(W_0 \tilde{s}) \). For instance, in case (i) we have \( a \ln W_0 + a E(\ln \tilde{r}) + cW_0^bE(\tilde{r}^b) + k = a \ln W_0 + a E(\ln \tilde{s}) + cW_0^bE(\tilde{s}^b) + k \), implying \( W_0 = \{aE(\ln \tilde{s}) - aE(\ln \tilde{r})\}^{1/b}/\{cE(\tilde{r}^b) - cE(\tilde{s}^b)\}^{1/b} \) if the gambles have been designated \( \tilde{r} \) and \( \tilde{s} \) such that both bracketed expressions are positive; if the two gambles are such that the bracketed expressions have opposite signs, then for these gambles no switch point exists.

**Corollary 2.1:** The only distinct multiplicatively one-switch (but not zero-switch) utilities \( U \) with \( U' > 0 \) and \( U'' < 0 \) for all \( W > 0 \) are \( a \ln W + W^b/b \ (b \not= 1, b \geq 0, a > 0) \) and \( (W^b/b) + cW^d \).
(b\(\tilde{0}\), cd>0, d<b\#1).

Each of these has DARA globally. Note that the utilities in Corollary 2.1 include the linear-plus-log and the linear-plus-power functions as particular cases (with b=1).

**Corollary 2.2:** There are no utilities \(U\) with \(U' > 0\) and \(U'' < 0\) for all \(W > 0\) which are one-switch both additively and multiplicatively.

The import of this corollary is that, if both additively and multiplicatively one-switch behavior are considered to be reasonable, a decision-maker must choose between them in choosing a utility function that best represents his preferences, because the two properties are mutually inconsistent under expected utility preferences.

It is natural to ask for what utility functions will preference switches occur only in favor of the portfolio with higher per-dollar mean return.

**Definition:** A utility function which allows switching between preference for \(W_0 \tilde{r}\) and \(W_0 \tilde{s}\) (with \(E\tilde{r} > E\tilde{s}\)) as \(W_0\) rises is referred to as *multiplicative-risk consistent* if, whenever such a preference reversal occurs, it is \(W_0 \tilde{r}\) which is preferred at values of \(W_0\) immediately above the switch value.

Note that Bell (1988) discussed the corresponding property for additive risks \(W_0 + x\tilde{r}\) and \(W_0 + y\tilde{s}\), referring to it as risk consistency.

**Theorem 3:** The only distinct infinitely differentiable utility functions \(U\) with \(U' > 0\) and \(U'' < 0\) for all \(W > 0\) which are multiplicative-risk consistent are \(U(W) = W + a \ln W\) (\(a > 0\)) and \(U(W) = W + cW^d\) (\(cd > 0\), \(d < 1\)).

**Proof:** Clearly multiplicative-risk consistent behavior implies the one-switch property, since
two switches would involve one to the higher-mean risk and one back to the lower-mean risk (or vice versa), one of which switches would violate the condition of switching only to a higher-mean risk.

Thus by Corollary 2.1 we must have \( U(W) = a \ln W + W^{b/b} \) (\( a > 0, \ b \neq 1, \ b \neq 0 \)) or \( U(W) = (W^b/b) + cW^d \) (\( b = 0, \ cd > 0, \ d < b \ # 1 \)).

Now consider \( EU(W_0 \tilde{r}) - EU(W_0 \tilde{s}) \) for \( U(W) = a \ln W + W^{b/b} \). With \( E\tilde{r} > E\tilde{s} \), a violation of the consistency property would entail \( d[EU(W_0 \tilde{r}) - EU(W_0 \tilde{s})]/dW_0 < 0 \) when evaluated at the switch value of \( W_0 \), which satisfies \( EU(W_0 \tilde{r}) = EU(W_0 \tilde{s}) \). The comparative static inequality implies \( [E\tilde{r}^b - E\tilde{s}^b] < 0 \), while there must exist a switch value \( W_0 \) satisfying \( aE(\ln \tilde{r} - \ln \tilde{s}) + W_0^{b/b}[E\tilde{r}^b - E\tilde{s}^b] = 0 \). Regardless of the sign of \( b \), this equality combined with the previous inequality requires \( bE(\ln \tilde{r} - \ln \tilde{s}) > 0 \). Thus there exists a violation of the consistency property if and only if there exist a pair of random variables \( \tilde{r}, \tilde{s} \) satisfying all of (i) \( E\tilde{r} > E\tilde{s} \), (ii) \( E\tilde{r}^b < E\tilde{s}^b \), and (iii) \( b E \ln \tilde{r} > b E \ln s \).

With \( b = 1 \) this is impossible since (i) and (ii) contradict each other; thus the utility function \( U(W) = a \ln W + W^b \) possesses the consistency property. But for any \( b < 1 \) (\( b \neq 0 \)), Appendix 1 shows that there exist \( \tilde{r}, \tilde{s} \) satisfying (i), (ii), and (iii), thus showing that the consistency property is not exhibited by \( a \ln W + W^b \) if \( b < 1 \) (\( b \neq 0 \)).

Now consider \( EU(W_0 \tilde{r}) - EU(W_0 \tilde{s}) \) for \( U(W) = (W^b/b) + cW^d \). With \( E\tilde{r} > E\tilde{s} \), again a violation of the multiplicative-risk consistency property would entail \( d[EU(W_0 \tilde{r}) - EU(W_0 \tilde{s})]/dW_0 < 0 \) when evaluated at \( W_0 \) satisfying \( EU(W_0 \tilde{r}) = EU(W_0 \tilde{s}) \). This equality requires \( W_0^{b-d} E(\tilde{r}^b - \tilde{s}^d) = -cbE(\tilde{r}^d - \tilde{s}^d) \). The comparative static inequality requires \( W_0^{b-d} E(\tilde{r}^b - \tilde{s}^d) < 0 \); substituting from the equality, this inequality becomes \( c(d - b)E(\tilde{r}^d - \tilde{s}^d) < 0 \) or, using the prior condition that \( d < b \), (iv) \( cE(\tilde{r}^d - \tilde{s}^d) > 0 \). Thus there exists a violation of the consistency property if
and only if there exist a pair of random variables $\tilde{r}$, $\tilde{s}$ satisfying (iv) as well as (v) $E\tilde{r} > E\tilde{s}$ and the condition that the switch value of $W_0$ exist, which using (iv) becomes (vi) $bE(\tilde{r}^b - \tilde{s}^b) < 0$. Now if $b = 1$ then (v) and (vi) are mutually inconsistent; so the $b = 1$ case--$U(W) = W + cW^d$--exhibits the consistency property. But Appendix 1 shows that for any $b < 1 (b \neq 0)$, there exist $\tilde{r}$, $\tilde{s}$ satisfying (iv), (v), and (vi), thus showing that the consistency property is not exhibited by $(W^b/b) + cW^d$ if $b < 1 (b \neq 0)$. QED.

3. Risk-Return Preferences and a Measure of Relative Risk

Bell (1988, 1995) showed that the linear-plus-exponential utility function has the feature that expected utility can be written as a function of mean final wealth and “risk” of the final wealth distribution: $EU(\tilde{W}) = E[a\tilde{W} - e^{-b\tilde{W}}] = a(E\tilde{W}) - e^{-b(E\tilde{W})}R_a$, where $R_a = E[e^{-b(\tilde{W} - EW)}] = \text{absolute risk}$. This measure of risk is specific to the individual, since it depends on the utility parameter $b$. It has the features that a mean-preserving spread always increases risk, and that a fixed absolute increment to final wealth in all states of nature—that is, a pure mean-shift of the distribution—leaves risk unaffected.

Such a measure of risk is appropriate for situations in which gambles are additive. Sarin and Weber (1993, pp. 142-3) presented an analogous measure of risk which also allows expected utility, for a particular family of utility functions, to be written as a function of mean wealth and risk; this measure of risk, which we adapt below, is relative and is appropriate for the multiplicative situations considered in this paper, in which final wealth $\tilde{W}$ equals initial wealth $W_0$ times portfolio return $\tilde{r}$. Just as the above measure of absolute risk applies in the context of additive-risk consistent utility functions, this measure of relative risk applies in the context of the utility functions identified in Theorem 3 as being
multiplicative-risk consistent.

By Theorem 3, the multiplicative-risk consistent one-switch utility functions are (i') \( W + a \ln W \) \((a > 0)\) and (ii') \( W + cW^d \) \((cd > 0, d < 1)\). We shall rescale the former by multiplying by \( k_1 \)
\((=1/a)\): (i') \( k_1W + \ln W \) \((k_1>0)\); and we shall rescale the latter by multiplying by \( k_2 \)
\((=1/cd)\): (ii') \( k_2W + (W^d/d) \) \((k_2 > 0, d < 1, d \neq 0)\). As always with power or log utilities, we assume that all realizations give positive wealth.

For case (i), with \( U(W) = k_1W + \ln W \) \((k_1>0)\), we can write expected utility as
\[
(1) \quad EU(\tilde{W}) = k_1(E\tilde{W}) + \ln (E\tilde{W}) - R_r, \quad R_r = E[-\ln (\tilde{W}/E\tilde{W})].
\]
For case (ii), with \( U(W) = k_2W + (W^d/d) \), \( k_2 > 0, d < 1, d \neq 0 \), we can write expected utility as
\[
(2) \quad EU(\tilde{W}) = k_2(E\tilde{W}) - (1/d)(E\tilde{W})^d[R_r-(d/d)\{(E\tilde{W})^d/E\tilde{W}\}] + (d/d). \quad R_r = (-d/d)(E\tilde{W})^d/(E\tilde{W}) + (d/d).
\]
Of course expression (2) could be written in slightly simplified form separately for the cases of \( d > 0 \) and \( d < 0 \).

Several things can be said about (1) and (2). First, each characterizes expected utility-based preferences in terms of mean \( E\tilde{W} \) and relative risk \( R_r \). Second, in each case expected utility is increasing in mean and decreasing in relative risk, as is appropriate for any reasonable concept of risk.

Third, in each case a riskfree situation is characterized by \( R_r = 0 \), and non-riskfree situations always have \( R_r > 0 \) (as can be seen from the concavity of \( \ln W \) and of \( W^d \) for \( 0 < d < 1 \), and from the convexity of \( W^d \) for \( d < 0 \)). Fourth, in each case every mean-preserving spread (MPS) is an increase in relative risk. This follows trivially from the known fact that expected utility falls due to a MPS; since \( E\tilde{W} \) is unchanged and expected utility is negatively affected by \( R_r \), it must be the case that the MPS increased \( R_r \). This effect of the MPS upon \( R_r \) can also be seen directly by again appealing to the
concavity of $\ln W$ and of $W^d$ for $0 < d < 1$, and the convexity of $W^d$ for $d < 0$.

The fifth observation about (1) and (2) is that relative risk is invariant to the multiplication of all realizations by a fixed positive constant. If, for example, we multiply all possible wealth realizations by, say, a constant greater than one, this causes the possible realizations to become farther apart from each other in an absolute sense, but not relative to their respective sizes. So, for example, if state 2 gives $g\%$ more wealth than state 1, then multiplying all realizations in all states by a fixed positive constant leaves state 2 still giving $g\%$ more wealth than state 1. It is in this relative sense that no change in risk has taken place.

Thus equations (1) and (2) provide one means of reconciling expected utility theory with the expression of preferences as a function of mean wealth and risk--one which lends itself naturally to a portfolio allocation context because the risk concept refers to relative risk. This way of expressing preferences applies for the utility functions which Theorem 3 identified as multiplicative-risk consistent.

The measures of relative risk in equations (1) and (2) lend themselves to a further result, analogous to the result in Theorem 3 that these utility functions only permit switches to higher-mean risks:

**Theorem 4:** The linear-plus-log and linear-plus-power utility functions associated with the relative risk expressions in equations (1) and (2) permit only switches to gambles with higher relative risk.

**Proof:** If $\tilde{r}$ and $\tilde{s}$ are such that one of them has higher or equal mean and lower relative risk than the other, then that one gives higher $EU$ for all $W_0$ (since by (1) or (2) $MEU / ME\tilde{W} > 0$ and $MEU / MR < 0$). So there can be no switch between $\tilde{r}$ and $\tilde{s}$ in this case. Thus any switch must be between
two risks one of which has higher mean and higher relative risk, in which case by Theorem 3 the switch is to the risk with higher mean and hence higher relative risk. **QED.**

4. Comparative Statics of $n$-Asset Portfolios

While Section 2 considered the nature of choices between two given portfolios, and how those choices vary with initial wealth, we now consider the effect of initial wealth upon the nature of $n$-asset portfolios in which the weights on the assets are continuous variables which are chosen optimally. As discussed in the introductory section, very few $n$-asset portfolio comparative static results exist in the literature, but the ones we obtain here are of course not precluded by the impossibility theorem of Hart (1975).

Our first result is that a rise in initial wealth always leads to a rise (or no change) in the mean return per dollar invested if and only if utility has precisely the form that Theorem 3 identified as necessary and sufficient for multiplicative-risk consistency in the context of a discrete choice between given portfolios.

**Theorem 5:** In optimal portfolios of $n$ assets (one of which may have a degenerate distribution at a single point, making it riskfree), a rise in initial wealth always causes a rise or no change in mean return per dollar invested if and only if the utility function, which is assumed to be increasing and concave for all $W > 0$, has the form (i) $U(W) = k_1 W + \ln W$ with $k_1 \geq 0$, or (ii) $U(W) = k_2 W + (W^d/d)$ with $k_2 \geq 0$ and $d < 1$, $d \neq 0$.

**Proof:** Sufficiency: If, for any $n$, some $U$ permits $d(E\tilde{W}/W_0)/dW_0 < 0$, then there exist some $W_{0,a}$ and $W_{0,b}$ with $W_{0,a} < W_{0,b}$ such that at $W_{0,a}$ $U$ chooses an optimal portfolio with final
wealth expressible as \( W_{0,a} \tilde{r} \) and at \( W_{0,b} \) \( U \) chooses an optimal portfolio with final wealth expressible as \( W_{0,b} \tilde{s} \) where \( E\tilde{r} > E\tilde{s} \). Thus \( U \) preferred \( W_{0,a} \tilde{r} \) over \( W_{0,a} \tilde{s} \), and as \( W_0 \) rose to \( W_{0,b} \) \( U \) switched preference, preferring \( W_{0,b} \tilde{s} \) over \( W_{0,b} \tilde{r} \). Since \( U \) switched preference to the gamble with lower \( (E\tilde{W} / W_0) \), by Theorem 3 \( U \) cannot be of the linear-plus-log or linear-plus-power form. Thus these utility forms are sufficient to always give \( d(E\tilde{W} / W_0) / dW_0 \neq 0 \).

**Necessity:** For a utility function \( U \) to give the indicated comparative static result in all circumstances, it must do so when there are two assets, with returns \( \tilde{r}_1 \) and \( \tilde{r}_2 \) and with \( E\tilde{r}_1 > E\tilde{r}_2 \). In this case the problem is \( \text{Max}_\alpha EU(W_0[\alpha \tilde{r}_1 + (1-\alpha)\tilde{r}_2] \), the first-order condition (with \( W_0 \) divided out) is \( E[U'(\tilde{W})(\tilde{r}_1 - \tilde{r}_2)] = 0 \), and the comparative static inequality is \( E[U''(\tilde{W})(\tilde{r}_1 - \tilde{r}_2)] [\alpha \tilde{r}_1 + (1-\alpha)\tilde{r}_2 \]$\# 0 \) which can be rewritten as follows by multiplying through by \(-W_0\):

\[
(3) \quad E[-W U''(\tilde{W})(\tilde{r}_1 - \tilde{r}_2)] \neq 0 ,
\]

where \( \tilde{W} = W_0[\alpha^* \tilde{r}_1 + (1-\alpha^*)\tilde{r}_2] \) with \( \alpha^* \) being the optimal value of \( \alpha \). Now \(-WU''(W) / V'(W)\) can be viewed as the derivative of a different function \( V \). Appendix 2 shows that if \( E[V'(\tilde{W})(\tilde{r}_1 - \tilde{r}_2)] \neq 0 \) at \( \alpha^* \) for all \( \tilde{r}_1, \tilde{r}_2 \) with \( E\tilde{r}_1 > E\tilde{r}_2 \) then we can write \( V(W) / aW + gU(W) \) for some parameters \( a \) and \( g \). This equation implies \( V'(W) = a + gU'(W) \); combining this with the definitional equation \( V(W) = -WU'(W) \) gives the following differential equation for \( U \): \( a + gU' = -WU'' \). The only solutions of this for \( U \) are (i) and (ii) in the statement of this theorem, where (i) applies if \( g=1 \) (with \( k_1 = -a \)), and where (ii) applies if \( g \tilde{O} 1 \) (with \( k_2 = -a/g \) and \( d = 1-g \)). The necessity of the parameter restrictions stated in the theorem \( (k_1 \neq 0, k_2 \neq 0, d < 1) \) follows from the assumed increasing and concave nature of \( U \) for all \( W > 0 \). **QED.**

It is reasonable to refer to the class of utility functions identified in Theorem 5, which always
exhibit \( d(E\tilde{W}/W_0) / dW_0 \geq 0 \), as increasingly mean-seeking. Notice that all utility functions in this class have decreasing Arrow-Pratt relative risk aversion; but, as Cass and Stiglitz (1972, p. 345) also pointed out, the latter utility function feature is seen by Theorem 5 to be not sufficient to establish the sign of \( d(E\tilde{W}/W_0) / dW_0 \).

Section 3 showed that for the class of utility functions we have now identified in Theorem 5 as being necessary and sufficient for \( d(E\tilde{W}/W_0) / dW_0 \geq 0 \) in all portfolios, expected utility can be written as a function which is increasing in \( E\tilde{W} \) and decreasing in relative risk, \( R_r \), as defined in equations (1) and (2). Theorem 6 will now show that, for this utility class, as \( W_0 \) rises the relative risk of the optimal portfolio rises or remains unchanged as well.

**Theorem 6:** In optimal \( n \)-asset portfolios, if utility has one of the forms (i) \( U(W) = k_1 W + \ln W \) with \( k_1 \geq 0 \), or (ii) \( U(W) = k_2 W + (W^d/d) \) with \( k_2 \geq 0 \) and \( d < 1 \), \( d \geq 0 \), then \( dR_r / dW_0 \leq 0 \), where \( R_r \) is defined in equations (1) and (2) respectively for forms (i) and (ii).

**Proof:** If, for any \( n \), a utility function of the linear-plus-power or linear-plus-log form permitted \( dR_r / dW_0 < 0 \), then there would exist some \( W_{0,a} \) and \( W_{0,b} \) with \( W_{0,a} < W_{0,b} \) such that at \( W_{0,a} \) \( U \) chooses an optimal portfolio with final wealth expressible as \( W_{0,a} \tilde{r} \) and at \( W_{0,b} \) \( U \) chooses an optimal portfolio with final wealth expressible as \( W_{0,b} \tilde{s} \) where the relative risk of \( \tilde{s} \) is less than that of \( \tilde{r} \). Thus \( U \) preferred \( W_{0,a} \tilde{r} \) over \( W_{0,a} \tilde{s} \), and as \( W_0 \) rose to \( W_{0,b} \) \( U \) switched preference, preferring \( W_{0,b} \tilde{s} \) over \( W_{0,b} \tilde{r} \). Since \( U \) switched preference to the gamble with lower relative risk, by Theorem 4 \( U \) cannot be of the linear-plus-log or linear-plus-power form, thus establishing a contradiction. QED.  

Since Theorem 6 states that the indicated utility functions exhibit an increasing acceptance of relative risk as initial wealth rises, in the broad context of all \( n \)-asset portfolios, it is reasonable to refer
to these utility functions as having the property of broadly decreasing relative risk aversion.

5. Conclusion

This paper has considered utility functions exhibiting certain desirable properties in a context in which gambles are multiplied by initial wealth. It was shown that the utility functions with constant relative risk aversion are the only increasing, concave utilities which never switch preference between two such multiplicative gambles as initial wealth is varied. The only increasing, concave utilities which cannot switch preference more than once as initial wealth is varied are the log-plus-power and power-plus-power functions and their limiting cases, the log-plus-linear and power-plus-linear functions. Thus no utility function is one-switch in the contexts of both additive and multiplicative gambles. It was also shown that the log-plus-linear and power-plus-linear utility functions are the only ones which switch only to a higher-mean gamble; for these utility functions, expected utility can be written as a function increasing in mean wealth and decreasing in relative risk, with relative risk defined in such a way as to increase in response to any mean-preserving spread and to remain unchanged when all realizations of a gamble are multiplied by a fixed positive constant (so that relative magnitudes of realizations are unchanged). As initial wealth increases, these utility functions were shown to switch only to a gamble with higher relative risk.

Furthermore, the log-plus-linear and power-plus-linear utility functions were shown to be the only ones for which, in the context of $n$-asset portfolios, a rise in initial wealth always causes a rise (or no change) in the expected return per dollar invested in the optimal portfolio. In addition, for these
utility functions a rise in initial wealth always causes a rise (or no change) in the relative risk of the optimal portfolio.

An argument can be made that various of these behavioral features—the multiplicative one-switch feature of Theorem 2, the multiplicative-risk consistency feature of Theorem 3, the expression of preferences in terms of mean and relative risk, and the comparative static features of Theorems 5 and 6 that a rise in initial wealth causes a rise or no change both in expected per-dollar portfolio return and in the portfolio's relative risk—are intuitively and empirically appealing. These considerations make a good case that in many situations, both positive and prescriptive, researchers would do well to describe preferences in terms of utility functions which are log-plus-linear or power-plus-linear.
Appendix 1

This appendix proves the existence of random variables \( \tilde{r}, \tilde{s} \) satisfying the set of inequalities (i), (ii), or the set (iv), (v), and (vi), as required in the proof of Theorem 3. First consider (iv)-(vi): we need (iv) \( cE(\tilde{r}^d - \tilde{s}^d) > 0 \), (v) \( E\tilde{r} > E\tilde{s} \), and (vi) \( bE(\tilde{r}^b - \tilde{s}^b) < 0 \). We will first let \( E\tilde{r} = E\tilde{s} \) and find \( \tilde{r}, \tilde{s} \) satisfying (iv) and (vi); then (v) can also be satisfied by raising \( E\tilde{r} \) by an arbitrarily small amount. Expectations of Taylor expansions of \( \tilde{r}^b \) and \( \tilde{s}^b \) around their common mean \( E\tilde{r} \) give

\[
(E(\tilde{r}^b)) = \frac{1}{0}^4 \left( \frac{M_{r^b}}{M_{r^i}} \right) (1/i!) M_{ir}
\]

\[
(E(\tilde{s}^b)) = \frac{1}{0}^4 \left( \frac{M_{s^b}}{M_{s^i}} \right) (1/i!) M_{is}
\]

where the derivatives are evaluated at the common mean \( E\tilde{r} \) and where \( M_{ir} \right] E(\tilde{r}^2 - E\tilde{r}^2)^i \) and \( M_{is} \right] E(\tilde{s}^2 - E\tilde{s}^2)^i \), the \( i \)th moments of \( \tilde{r} \) and \( \tilde{s} \) respectively around the mean. In fashion analogous to Bell (1988), we choose \( \tilde{r} \) and \( \tilde{s} \) having \( M_{ir} = M_{is} \) for all \( i > 3 \), so the expressions for \( E\tilde{r}^b \) and \( E\tilde{s}^b \) differ only in the terms with \( i = 2, 3 \). Condition (vi) that \( b(E\tilde{r}^b - E\tilde{s}^b) < 0 \) is satisfied, regardless of the sign of \( b \), if

\[
3(E\tilde{r})(M_{2r} - M_{2s}) + (b - 2)(M_{3r} - M_{3s}) > 0.
\]

Likewise, replacing exponent \( b \) with \( d \) in (A1) and (A2), condition (iv) that \( cE(\tilde{r}^d - \tilde{s}^d) > 0 \) is satisfied, regardless of the sign of \( c \), if

\[
3(E\tilde{r})(M_{2r} - M_{2s}) + (d - 2)(M_{3r} - M_{3s}) < 0.
\]

Remembering that \( b < 1 \) and \( d < 1 \), (A3) and (A4) can be combined as

\[
3(E\tilde{r})(M_{2r} - M_{2s}) / (2 - b) + (3 - 2)(M_{3r} - M_{3s}) > 3(E\tilde{r})(M_{2r} - M_{2s}) / (2 - d).
\]

Thus a range exists for \( (M_{3r} - M_{3s}) \) if \( (M_{2r} - M_{2s}) > 0 \) (since it is given that \( d < b \)). Thus random variables \( \tilde{r} \) and \( \tilde{s} \) satisfying (A5) and with \( M_{ir} = M_{is} \) for \( i > 3 \) will satisfy (iv) and (vi), and if \( E\tilde{r} \) is
raised infinitessimal above $E\tilde{s}$ then (v) is satisfied as well.

Now consider (i)-(iii): we need (i) $E\tilde{r} > E\tilde{s}$, (ii) $E\tilde{r}^b < E\tilde{s}^b$, and (iii) $b E \ln \tilde{r} > b E \ln \tilde{s}$.

Again tentatively letting $E\tilde{r} = E\tilde{s}$, taking expectations of Taylor expansions of $\tilde{r}^b$, $\tilde{s}^b$, $\ln \tilde{r}$, and $\ln \tilde{s}$, we find that if $b > 0$ condition (ii) implies (A3) again and condition (iii) implies

(A6)  \[ 3(E\tilde{r})(M_{2r} - M_{2s}) - 2(M_{3r} - M_{3s}) < 0. \]

Combining (A3) and (A6) gives an inequality chain identical to (A5) but with $d$ set equal to zero.

Alternatively, if $b < 0$ conditions (ii) and (iii) imply (A3) and (A6) with the inequalities reversed, and these can be combined to give (A5) with the inequalities reversed and with $d$ set equal to zero. In either case ($b > 0$ or $b < 0$), if $(M_{2r} - M_{2s})$ is chosen to be positive a range exists for $(M_{3r} - M_{3s})$.

Thus random variables $\tilde{r}$ and $\tilde{s}$ satisfying the indicated inequalities will satisfy (ii) and (iii), and if $E\tilde{r}$ is raised infinitessimally above $E\tilde{s}$ then (i) is satisfied as well.

**Appendix 2**

This appendix shows that, as required for the proof of Theorem 5, if $\hat{a}^*$ solves $\text{Max}_{\hat{a}}$

$EU(W_0[\hat{a}\tilde{r}_1 + (1-\hat{a})\tilde{r}_2])$ where $E\tilde{r}_1 > E\tilde{r}_2$ and if $E[V(W_0[\hat{a}^*\tilde{r}_1 + (1-\hat{a}^*)\tilde{r}_2])(\tilde{r}_1 - \tilde{r}_2)] \neq 0$ in all such problems, then $V(W) \sim aW + gU(W)$ for some parameters $a, g$. First, if we had $E\tilde{r}_1 < E\tilde{r}_2$, then the required inequality would be that $E[V(W_0[\hat{a}^*\tilde{r}_1 + (1-\hat{a}^*)\tilde{r}_2])(\tilde{r}_1 - \tilde{r}_2)]$ $\neq 0$. So by continuity we must have that, if $E\tilde{r}_1 = E\tilde{r}_2$, $E[V(W_0[\hat{a}^*\tilde{r}_1 + (1-\hat{a}^*)\tilde{r}_2])(\tilde{r}_1 - \tilde{r}_2)] = 0$. This must hold for any joint distribution of $\tilde{r}_1$, $\tilde{r}_2$, including this one: $Pr(\tilde{r}_1 - \tilde{r}_2) = -1$, $\tilde{r}_2 = r_{21}$] = p, $Pr(\tilde{r}_1 - \tilde{r}_2) = 1$, $\tilde{r}_2 = r_{22}$] = 1/2, $Pr(\tilde{r}_1 - \tilde{r}_2) = -1$, $\tilde{r}_2 = r_{23}$] = 1/2 - p, for parameters $r_{21} < r_{22} < r_{23}$. Here $E(\tilde{r}_1 - \tilde{r}_2)$ $= 0$.

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Now fix $r_{21}$ and $r_{23}$, and choose $r_{22}$ between $r_{21}$ and $r_{23}$, and choose $p$ such that $E\{U'(r_{21}) \cdot (r_{21} - \tilde{r}_2 - r_{22})\} = 0$ (implying that $\hat{a}=0$ is the optimum for $U$). Hence $(1/2)U'(r_{22}) - pU'(r_{21}) + (p-1/2)U'(r_{23}) = 0$. Since $U'$ is monotonically decreasing the three fixed values of $U'$ have $U'(r_{23}) < U'(r_{22}) < U'(r_{21})$. Let $h$ be such that $V'(W) / h(U'(W))$ for all $W$. Since $E\tilde{r}_1 = E\tilde{r}_2$, as shown above we have that $E[h(U'(\tilde{W})) (\tilde{r}_1 - \tilde{r}_2)] = 0$, so $(1/2)h(U'(r_{22})) - ph(U'(r_{21})) + (p-1/2)h(U'(r_{23})) = 0$ and thus $h(U'(r_{22})) = \hat{a}h(U'(r_{21})) + (1-\hat{a})h(U'(r_{23}))$ where $\hat{a}=2p$. Hence $(U'(r_{22}), h(U'(r_{22})))$ must lie on a line connecting $(U'(r_{21}), h(U'(r_{21})))$ and $(U'(r_{23}), h(U'(r_{23})))$; so $h$ is linear on the interval $(U'(r_{23}), U'(r_{21}))$. Since the endpoints of this interval could be arbitrarily chosen, and this analysis applies for all such intervals, $h$ is everywhere linear. Thus there exist $a,g$ such that $h(U'(r_2)) = a + gU'(r_2)$. Therefore $V' = a + gU'$ and so $V(W) = aW + gU(W) + c$, where $c$ is an irrelevant constant that can be dropped.
REFERENCES


Bell, David E. "One-Switch Utility Functions and a Measure of Risk," Management Science 34, December 1988, 1416-1424.


NOTES

1 A few other results exist for n-asset portfolios under joint distributional restrictions: Cass and Stiglitz (1972) and Mitchell (1994) consider joint distributions in which the number of states of nature equals the number of assets.

2 One can analogously use Bell's (1988) Proposition 9 and the transformation $U/V^B ln$ to show that an infinitely differentiable utility function is multiplicatively n-switch if and only if it can be written as $f_0(lnW) + \sum_{i=1}^{k} f_i(ln W)W^i$, where $f_i(.)$ is a polynomial of order $n_i$ such that $\sum_{i=0}^{k} n_i \# n+1-k$.

3 An alternative approach to proving sufficiency in Theorem 5 and to proving Theorem 6, without relying on the multiplicative-risk consistency feature of Theorem 4, is as follows: First use simple calculus for the $n=2$ case to show that for these utility functions a rise in $W_0$ gives a rise or no change in the share in the higher-mean asset and thus a rise or no change in $EW/W_0$ and $R_r$. Then show, analogously to Mitchell and Douglas (1997), that if there is a portfolio environment with $n>2$ permitting $d(EW/W_0)/dW_0 < 0$ or $dR_r/dW_0 < 0$ then there is also an $n=2$ portfolio environment permitting the same, which contradicts what was just established.

4 Of course, if $k_1 = 0$ or $k_2 = 0$ in the respective utility functions, $W_0$ drops out of the portfolio first-order conditions, and so $W_0$ has no effect on any optimal portfolio weights. Hence $d(EW/W_0)/dW_0 = 0$ and $dR_r/dW_0 = 0$.

5 Given a random variable $\tilde{r}$ with its infinite sequence of moments, we can specify another sequence of moments identical for all $M_i \geq 3$ and with $M_2$ and $M_3$ both infinitesimally lower and in accordance with (A5) below; by continuity this new sequence itself satisfies the sufficient conditions...
(Widder 1946, ch. 3) for existence of an associated random variable \( \tilde{s} \).
SWITCHING, MEAN-SEEKING, AND RELATIVE RISK
WITH TWO OR MORE RISKY ASSETS

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and
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This paper relates and extends two strands of literature on behavioral implications of utility function features. A class of utility functions switching preference no more than once between $W_0 \tilde{r}$ and $W_0 \tilde{s}$ is identified ($\tilde{r}$ and $\tilde{s}$ being per-dollar returns of alternative portfolios), as is the subclass permitting only switches to the higher-mean portfolio. This latter subclass, which permits expected utility to be expressed in terms of the mean and suitably defined relative risk of final wealth, never decreases expected per-dollar final wealth and never decreases relative risk of final wealth in optimal $n$-asset portfolios as initial wealth rises.

**Keywords:** Utility functions, relative risk, one-switch utility, multi-asset portfolios, mean-seeking, risk-value models

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