ORTHOGONALIZED EQUITY RISK PREMIA

AND

SYSTEMATIC RISK DECOMPOSITION

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Abstract

To solve the dependency problem between factors, in the context of linear multi-factor models, this study proposes an optimal procedure to find orthogonalized risk premia, which also facilitates the decomposition of the coefficient of determination. Importantly, the new risk premia may diverge significantly from the original ones. The decomposition of risk allows one to explicitly examine the impact of individual factors on the return variation of risky assets, which provides discriminative power for factor selection. The procedure is experimentally robust even for small samples. Empirically we find that even though on average, approximately eighty (sixty-five) percent of style (industry) portfolios' volatility is explained by the *market* and *size* factors, other factors such as *value*, *momentum* and *contrarian* still play an important role for certain portfolios. The components of systematic risk, while dynamic over time, generally exhibit negative correlation between *market*, on one side, and *size*, *value*, *momentum* and *contrarian*, on the other side.

Keywords: Orthogonalization, Systematic Risk, Decomposition, Fama-French Model, Asset Pricing.

JEL Classification: G11, G12, G14

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I. Introduction

Under the traditional single-factor Sharpe (1964) and Lintner (1965) Capital Asset Pricing Model (CAPM), the market beta captures a stock's systematic risk for all rational, risk-averse investors. Therefore, a decomposition of the market beta is sufficient to break down the systematic risk of a stock.[1] For example, Campbell and Vuolteenaho (2004) break the market beta of a stock into a 'bad' component that reflects news about the market's future cash flows and a 'good' component that reflects news about the market's discount rates. In an earlier paper, Campbell and Mei (1993) show that the market beta can be decomposed into three sub-betas that reflect news about future cash flows, future real interest rates and a stock's future excess returns, respectively. Acharya and Pedersen (2005) develop a CAPM with liquidity risk by separating the market beta of a stock into four sub-betas that reflect the impact of illiquidity costs on the systematic risk of an asset. In many cases, the analysis of decomposed market beta has been applied to examine the size and/or book-to-market anomalies. Although the beta-decomposition is useful to describe the structure and source of systematic variation of returns on risky assets, it is complicated under multi-factor frameworks.[2]

The purpose of this paper is to develop an optimal procedure to identify the underlying uncorrelated components of common factors by a simultaneous orthogonal transformation of sample data, such that the systematic variation of stock returns becomes decomposable.

In the past two decades, one of the most extensively studied areas of financial research has concentrated on alternative common risk factors, in addition to the market risk premium, that could characterize the cross-section of stock expected returns. Fama and French (1992, 1993, 1996, 1998) document that the company's market capitalization - *size* and the company's *value*,

[1] According to the CAPM, the systematic risk is measured by $(\beta_j \sigma_{RM})^2$. Since the market risk premium is the only factor faced by all investors, β_j is sufficient to determine the systematic risk.

[2] Campbell and Mei (1993) show that the complication is due to the possible covariance between the risk price of one factor and the other factors.

which is characterized by ratios of book-to-market (B/M), earnings to price (E/P) or cash flows to price (C/P), together predict the return on a portfolio of stocks with much higher accuracy than the market beta and the traditional CAPM.[3] In addition to the *size* and *value* effects, Jegadeesh and Titman (1993, 2001), Rouwenhorst (1998), Chan, Jegadeesh and Lakonishok (1996) reveal that short-term past returns or past earnings predict future returns. Average returns on the best prior performing stocks (i.e. *winners*) exceed those of the worst prior performing stocks (i.e. *losers*), attesting the existence of *momentum* in stock prices. Conversely, DeBondt and Thaler (1985, 1987) reveal a *contrarian* effect such that stocks exhibiting low long-term past returns outperform stocks with high long-term past returns. DeBondt and Thaler (1985, 1987), Chopra, Lakonishok, and Ritter (1992) and Balvers et al. (2000) suggest a profitable contrarian strategy of buying the *losers* and shorting the *winners*.

Due to the common effects such as *size*, *value*, *momentum* and *contrarian*, additional factors, besides the *market risk premium*, must be considered for the determination of the return generating process for risky assets. Therefore, multi-factor market models have been widely employed by both financial academics and practitioners. Under the multi-factor framework, the expected excess return on a risky asset is specified as a linear combination of beta coefficients and expected premia of individual factors. Fama and French (1993) explicitly demonstrate that if there are multiple common factors in stock returns, they must be in the market return. This indicates that returns of common factors are correlated. Consequently, under the multi-factor framework, although the beta coefficient corresponding to an individual factor provides a sensitivity measure of an asset's return to the factor's variation, it may not catch precisely the systematic variation of the asset with respect to that factor. For instance, low beta might not indicate low systematic risk. The volatility of an asset's returns is determined jointly not only by

[3] Fama and French (1992, 1996, 1998) show that the investment strategy of buying the *small*/ *value* and shorting the *big*/ *growth* stocks produces positive returns.

the betas, but also by the variances and covariances of the factors' premia. Therefore, to provide a clear image of the separate roles of common factors in stock returns, determining the factors' underlying uncorrelated components becomes necessary.

This paper proposes an optimal simultaneous orthogonal transformation of sample returns on factors. The data transformation allows us to identify the underlying uncorrelated components of common factors. Specifically, the inherent components of factors retain their variances, but their cross-sectional covariances are zero. In addition, a multi-factor regression using the orthogonalized factors has the same coefficient of determination (R-square, i.e. the ratio of systematic variation to the overall volatility of a risky asset) as that using the original, non-orthogonalized factors. Since the coefficient of determination is a measure of the systematic risk of an asset, extracting the core, stand alone components of common factors enables us to decompose the systematic risk by disentangling the R-square, based on factors' volatility and their corresponding betas. Different from the Fama and French (1993) approach, our methodology is democratic, i.e. free of sequence bias.[4] Using Monte Carlo simulations, we demonstrate that the orthogonal transformation is robust, in that it produces unbiased estimates of the population systematic risk even for small samples. By applying our methodology to Kenneth French's data sets, we show empirically that the return variation of assets is now decomposable by factors.[5] We find that, generally, the *market*, *size* and *value* factors are the largest sources of systematic risk (not always in that order), while other factors such as

[4] Fama and French (1993, p27) clearly demonstrate that since the market return is a mixture of the multiple common factors, an orthogonalization of the market factor is necessary so that it can capture common variation in returns, left by other factors such as *size* or *value*. We argue that not only the market factor, but all factors need to be orthogonally transformed to eliminate the dependence problem among them. Although Fama-French's orthogonalization procedure for the market factor is straightforward, their method would involve selection of the leader (i.e. starting vector) and sequence biases, if one applied it to orthogonalize more than one factor in the model. Specifically, a different selection of the leader factor or a different orthogonalization sequence generates different transformation results.

[5] Kenneth French's Data Library is publicly accessible at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

momentum and *contrarian* play relatively small roles in stock volatility determination.

The paper is organized as follows. In Section II, after an illustration of systematic risk decomposition problems in multi-factor market models, we present our procedure of symmetric orthogonal transformation and risk-decomposition. Next, in Section III, we illustrate the procedure empirically, using sample data from Kenneth French's Data Library. The final section of the paper provides brief concluding remarks.

II. Orthogonalization Procedure

Empirically, suppose a risky asset *j*'s return generating process can be linearly determined by a set of *k*-common factors (returns or macro-variables, denoted by f_k) [e.g. the *market* (*RM)*, *size* (*SMB*), *value* (*HML*), *momentum* (*Mom)*, and *long-term reversal* (*Rev*) premia], as shown in the following regression model:

$$r_{j,t} = \alpha_j + \sum_{k=1}^{K} \beta_{k_{j,t}} f_{k,t} + \varepsilon_{j,t}, \tag{1}$$

where the residual term ε_j is assumed to be uncorrelated with f_k, but f_k are not independent from one another.[6]

The systematic variation of asset *j*'s returns can then be measured by:

$$\sigma_{sr_j}^2 = \sum_{l}^{K} \sum_{k}^{K} \beta_{k_j} \beta_{l_j} Cov(f_k, f_l), \tag{2}$$

while the coefficient of determination or *R*-square is the ratio of systematic variation to the total return variation ($\sigma_{sr_j}^2 / \sigma_j^2$).

It is important to note that under the multi-factor framework, the systematic risk depends not only on the beta coefficients but also on the factors' variance-covariance. Thus, beta coefficients alone are inappropriate measures for systematic variation associated with different factors. The purpose of this paper is to develop a decomposable systematic risk measure.

[6] For instance, the *market* factor is a hodgepodge of the multiple common factors, and the factor portfolios of *size*, *value*, *momentum* and *contrarian* are all formed using securities in the same market, and thus their returns are not uncorrelated.

As shown in (2), $\sigma^2_{sr_j}$ is not decomposable due to the mutual correlation between factors.[7] To eliminate the impact of factors' covariances, but maintain their variances, the underlying component of each factor's returns needs to be identified by an orthogonal transformation of sample returns.

II.A. Methodology

For convenience purposes, we present the transformation procedure in a matrix format. Let $\underset{T\times 1}{f^k} = \left[f_1^k, f_2^k, \cdots, f_T^k\right]'$ be the sample returns of the *k*-th factor for k = 1, 2, …, *K*, and $\underset{T\times K}{F} = \left[f_t^k\right]_{t=1,\ldots,T}^{k=1,\ldots,K}$ their corresponding *T* by *K* matrix.[8] Our purpose is to derive from $\underset{T\times K}{F}$ a matrix of mutually uncorrelated and variance-preserving vectors, denoted by $\underset{T\times K}{F^\perp} = \left[f_t^{k^\perp}\right]_{t=1,\ldots,T}^{k=1,\ldots,K}$, so that the systematic return variation can be estimated by the following decomposable form:

$$\hat{\sigma}^2_{sr_j} = \sum_{k=1}^{K}\left(\hat{\beta}^\perp_{k_j}\hat{\sigma}_{f^k}\right)^2 = \hat{\sigma}^2_j - \hat{\sigma}^2_{\varepsilon_j}\ , \tag{3}$$

where $\hat{\sigma}^2_{sr_j}$ is the estimate of $\sigma^2_{sr_j}$, $\hat{\beta}^\perp$ and $\hat{\sigma}^\perp$ are the estimates of beta and standard deviation from sample data after the orthogonal transformation, and $\hat{\sigma}^2_j$ and $\hat{\sigma}^2_{\varepsilon_j}$ are the estimated variance of asset *j*'s returns and its residual variance, respectively. Importantly, both the intercept and the error term in equation (1) stay the same after transformation.

To obtain $\underset{T\times K}{F^\perp}$, we employ a methodology attributed to Löwdin (1970) - in the quantum chemistry literature, and to Schweinler and Wigner (1970) – in the wavelet literature. This is a *democratic* procedure as opposed to a *sequential* approach that is sensitive to the order in which

[7] For simplicity and without losing generality, henceforth we will only refer to common factors as *factor portfolio returns*, but the model can also be applied to other systematic factors (e.g. macro-economic variables).

[8] The factors are assumed to be linearly independent (i.e. $F_{T\times K}$ is full rank, where *T* is the number of time periods, and *K* is number of factors).

vectors are selected.[9] This distinctive characteristic is essential for a proper decomposition, as we need to treat all the factors on an equal footing. Thus, the orthogonal transformation of all factors has to be conducted jointly and simultaneously.[10]

We choose the *symmetric* form of orthogonalization, which minimizes the overall difference between the original and the orthogonal vectors, thus maximizing the resemblance between the two sets of data.[11] We apply it to the demeaned original factors, which ensures that the resulting vectors are not only mathematically orthogonal, but also uncorrelated.

Let $\overleftrightarrow{F}_{T\times K} = \left[\overleftrightarrow{f}_t^k\right]_{t=1,\ldots,T}^{k=1,\ldots,K} = \left[f_t^k - \overline{f}^k\right]_{t=1,\ldots,T}^{k=1,\ldots,K}$ be a demeaned matrix of $F_{T\times K}$. We define a linear transformation $S_{K\times K}$ of the set $\overleftrightarrow{F}_{T\times K}$ to $\overleftrightarrow{F}_{T\times K}^{\perp}$, as follows:

$$\overleftrightarrow{F}_{T\times K}^{\perp} = \overleftrightarrow{F}_{T\times K} S_{K\times K}\,. \tag{4}$$

To obtain $S_{K\times K}$ (and then $\overleftrightarrow{F}_{T\times K}^{\perp}$), the first step is to calculate the variance-covariance matrix of the factors sample returns $\left(\Sigma_{K\times K}\right)$, and take $M_{K\times K} = (T-1)\Sigma_{K\times K}$, that is:

$$M_{K\times K} = \begin{bmatrix} (\overleftrightarrow{f}^1)'(\overleftrightarrow{f}^1) & (\overleftrightarrow{f}^1)'(\overleftrightarrow{f}^2) & \cdots & (\overleftrightarrow{f}^1)'(\overleftrightarrow{f}^K) \\ (\overleftrightarrow{f}^2)'(\overleftrightarrow{f}^1) & (\overleftrightarrow{f}^2)'(\overleftrightarrow{f}^2) & \cdots & (\overleftrightarrow{f}^2)'(\overleftrightarrow{f}^K) \\ \vdots & \vdots & \vdots & \vdots \\ (\overleftrightarrow{f}^K)'(\overleftrightarrow{f}^1) & (\overleftrightarrow{f}^K)'(\overleftrightarrow{f}^2) & \cdots & (\overleftrightarrow{f}^K)'(\overleftrightarrow{f}^K) \end{bmatrix}. \tag{5}$$

The matrix $\overleftrightarrow{F}_{T\times K}^{\perp}$ will be orthonormal if

$$\left(\overleftrightarrow{F}_{T\times K}^{\perp}\right)'\overleftrightarrow{F}_{T\times K}^{\perp} = \left(\overleftrightarrow{F}_{T\times K} S_{K\times K}\right)'\overleftrightarrow{F}_{T\times K} S_{K\times K} = S'_{K\times K}\left(\overleftrightarrow{F}'_{T\times K}\overleftrightarrow{F}_{T\times K}\right)S_{K\times K} = S'_{K\times K} M_{K\times K} S_{K\times K} = I_{K\times K} \tag{6}$$

[9] The classical sequential orthogonalization procedure is the Gram-Schmidt process. For a comparison between the two approaches, see for instance Chaturvedi, Kapoor and Srinivasan (1998) and Löwdin (1970).
[10] Compared to orthogonal rotation techniques employed in factor analysis (e.g. *varimax*, *quartimax* or *equamax*), our procedure has the advantage of providing a bijective transformation of the original variables (i.e. a one-to-one and onto correspondence between the original and the orthogonally-transformed sets).
[11] For an elaborate explanation, see Srivastava (2000) and Löwdin (1970).

or equivalently,

$$\underset{K\times K}{S}\,\underset{K\times K}{S'} = \underset{K\times K}{M^{-1}}\,. \tag{7}$$

The general solution of equation (7) is $\underset{K\times K}{S} = \underset{K\times K}{M^{-1/2}}\underset{K\times K}{C}$, where C is an arbitrary orthogonal matrix. For $\underset{K\times K}{C} = \underset{K\times K}{I}$, where $\underset{K\times K}{I}$ is the identity matrix, the orthogonalization procedure is called *symmetric*. To be able to calculate $S_{K\times K}$, we identify an orthogonal matrix $O_{K\times K}$ (i.e. $O'_{K\times K} = O^{-1}_{K\times K}$) that brings $M_{K\times K}$ to a diagonal form $D_{K\times K}$ (i.e. $O^{-1}_{K\times K}M_{K\times K}O_{K\times K} = D_{K\times K}$).[12] Thus, $M_{K\times K}$ can be factorized as:

$$\underset{K\times K}{M} = \underset{K\times K}{O}\,\underset{K\times K}{D}\,\underset{K\times K}{O^{-1}}\,,^{13} \tag{8}$$

where the k-th column of $O_{K\times K}$ is the k-th *eigenvector* of the matrix $M_{K\times K}$, and $D_{K\times K}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues (λ), i.e., $D_{kk} = \lambda_k$, where k goes from 1 to K.

Solving for $S_{K\times K}$ from equations (7) and (8), we obtain the symmetric matrix:

$$\underset{K\times K}{S} = \underset{K\times K}{O}\,\underset{K\times K}{D^{-1/2}}\,\underset{K\times K}{O'}\,, \tag{9}$$

$$\text{where: } \underset{K\times K}{\mathrm{D}^{-1/2}} = \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{\lambda_K} \end{bmatrix}. \tag{10}$$

[12] If $\underset{K\times K}{C} = \underset{K\times K}{O}$ instead, the orthogonalization is termed *canonical*. This form is not appropriate in our case, as it does not maintain the resemblance with the original data.

[13] Equation (7) could also be solved using the Cholesky factorization (often employed, for instance, in the Generalized Least Squares estimation), but the procedure would produce orthogonal factors whose values depend on their sequence (i.e. the algorithm would not be democratic). Also note that the GLS estimation, different from our procedure, transforms not only the independent variables, but also the dependent variable and the error term.

Finally, we rescale the factors to the original variance, using the following transformation:

$$S_{K\times K} \mapsto S_{K\times K}\sqrt{T-1}\begin{bmatrix}\sigma_1 & 0 & \cdots & 0\\ 0 & \sigma_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sigma_K\end{bmatrix}, \tag{11}$$

where σ_i represents the standard deviation of factor i, for i = 1, 2, …, K.

Hence, the matrix $S_{K\times K}$ [as transformed in equation (11)], when substituted in equation (4), gives the symmetric orthogonal transformation of the demeaned factor-matrix $\overleftrightarrow{F}_{T\times K}$.

To obtain $F^{\perp}_{T\times K}$, we perform the following straight-forward transformation:

$$\overleftrightarrow{F}^{\perp}_{T\times K} + \mathbf{1}_{T\times 1}\overline{F}_{1\times K}S_{K\times K} = \overleftrightarrow{F}_{T\times K}S_{K\times K} + \mathbf{1}_{T\times 1}\overline{F}_{1\times K}S_{K\times K} = \left(\overleftrightarrow{F}_{T\times K} + \mathbf{1}_{T\times 1}\overline{F}_{1\times K}\right)S_{K\times K} = F_{T\times K}S_{K\times K} = F^{\perp}_{T\times K}, \tag{12}$$

where $\mathbf{1}_{T\times 1}$ is a vector of ones and $\overline{F}_{1\times K}$ is the mean of $F_{T\times K}$.

Hence, the matrix $F^{\perp}_{T\times K}$ is a conversion of matrix $\overleftrightarrow{F}^{\perp}_{T\times K}$, not an orthogonalized matrix per se. However, considering that adding constant terms to orthogonal vectors results in uncorrelated vectors, we can refer to $F^{\perp}_{T\times K}$, for simplicity, as the orthogonalized matrix of $F_{T\times K}$.

To be able to understand what matrix $S_{K\times K}$ (or its inverse) represents, we can write each factor f^k, where k = 1, 2, …, K, as:

$$f^k = \psi_{1k}f^{1^{\perp}} + \psi_{2k}f^{2^{\perp}} + \cdots + \psi_{Kk}f^{K^{\perp}}, \tag{13}$$

where the coefficients ψ_{lk} are obtained through the orthogonalization procedure described above (i.e. they are the elements of the inverse of matrix $S_{K\times K}$, in its final form).

If we calculate the covariance between f^k and $f^{k^{\perp}}$, we obtain that

$$\mathrm{cov}\left(f^k, f^{k^{\perp}}\right) = \psi_{kk}\times \mathrm{var}\left(f^{k^{\perp}}\right) \Rightarrow \psi_{kk} = \frac{\mathrm{cov}\left(f^k, f^{k^{\perp}}\right)}{\mathrm{var}\left(f^{k^{\perp}}\right)} = \frac{\mathrm{cov}\left(f^k, f^{k^{\perp}}\right)}{\sigma_k\sigma_k^{\perp}} = \mathrm{corr}\left(f^k, f^{k^{\perp}}\right) \tag{14}$$

Moreover, it can be shown (see the Appendix) that for any k and l, $\psi_{kl} = \text{corr}\left(f^k, f^{l^\perp}\right)$. Thus, the inverse of matrix $S_{K\times K}$ is the correlation matrix between the original and the orthogonal factors. So, if we consider $S_{K\times K} = \psi^{-1}_{K\times K}$, where $\psi_{K\times K} = \left[Corr(f^k, f^{l^\perp})\right]_{\substack{k=1,\dots,K\\ l=1,\dots,K}}$, the last equality in equation (12) can be rewritten as:

$$F^{\perp}_{T\times K} = F_{T\times K}\psi^{-1}_{K\times K}\,. \tag{15}$$

That is, the orthogonal factors are linear combinations of the original factors, with the coefficients taken from the inverse correlation matrix, between the original and the uncorrelated factors. Each orthogonal factor deviates from its original counterpart in such a way that the common variation is partitioned symmetrically.

To demonstrate the consistency between the decomposable systematic risk estimate, $\sum_{k=1}^{K}\left(\hat{\beta}^{\perp}_{k_j}\hat{\sigma}^{\perp}_{f^k}\right)^2$, and the systematic risk estimate from regression, $\hat{\sigma}^2_{sr_j} = \hat{\sigma}^2_j - \hat{\sigma}^2_{\varepsilon_j}$ as shown in equation (3), we note that the orthogonal transformation retains the original sum of squared errors (*SSE*) of equation (1) [i.e. *min* $\{\varepsilon'\varepsilon\}$]. Mathematically, the space generated by $F^{\perp}$ is the same, by definition, as the one generated by F.[14] This implies that $\left\{F\tilde{\beta}\middle|\tilde{\beta}\in\Re^5\right\} = \left\{F^{\perp}\tilde{\beta}^{\perp}\middle|\tilde{\beta}^{\perp}\in\Re^5\right\}$, meaning that all linear combinations of F span the same space as all linear combinations of $F^{\perp}$. Therefore, the range of the function of F [defined as $(r - F\tilde{\beta})'\,(r - F\tilde{\beta})$] is identical to that of the function of $F^{\perp}$ [defined as $(r - F^{\perp}\tilde{\beta}^{\perp})'\,(r - F^{\perp}\tilde{\beta}^{\perp})$]. Since the lower boundary of the two ranges is the same, *min* $\{\varepsilon'\varepsilon\}$ for F is identical to that for $F^{\perp}$.

[14] Moreover, by adding a column vector of ones to F and to $F^{\perp}$ (to account for intercepts), the two resulting spaces will still be identical.

II.B. Monte Carlo Simulations

Since the orthogonal transformation is a numerical data process, an examination of the sampling errors for the robustness of our estimation is important. Table I reports the mean squared errors (*MSE*) for a set of Monte Carlo simulations. We generate data for a five-factor linear model that has the following structure:

$$r = 0.02 + 1.2f_1 + 0.1f_2 + 0.3f_3 - 0.2f_4 + 0.7f_5 + e, \tag{16}$$

where f_k , $k = 1,2,\ldots5$, follow two hypothetical forms of distribution, multivariate-normal and multivariate-lognormal, respectively. The loadings in equation (16) are arbitrarily chosen so that they make sense economically. Different values will not cause significant changes in the robustness of our estimators. A non-zero covariance matrix of f_k is predetermined. In addition, the zero-mean residuals, e, follow by turns, one of the two processes: homoscedastic [white noise] and heteroscedastic [GARCH(1,1)].

[Insert Table I about here]

We calculate the *MSE* of decomposable systematic risk estimates for ten thousand trials. The random samples have, by turns, five different sizes: 50, 150, 300, 500, and 1,000. Table I shows that the *MSE* consistently and roughly proportionally decreases as the sample size increases. Specifically, the *MSE* drops by more than 80 percent when the sample size is increased from 50 to 300, and by more than 90 percent when increased from 50 to 500. This demonstrates that our estimate in equation (3) is robust, especially for sample sizes over 50.

In addition, we examine the robustness of *individual* decomposed systematic risk measures with respect to a set of five correlated factors in a hypothetical population with a finite number (twenty-five thousand) of outcomes. The uncorrelated components of the factors in the population are determined numerically by the orthogonal transformation. The population decomposed systematic risk for each factor $k, \left(\beta_k^{\perp}\sigma_{f_k}^{\perp}\right)^2$, is then calculated. Again, we generate ten-thousand random samples for each sample size (50, 150, 300, 500, and 1000 observations). Table II presents the *MSE* of individual decomposed measures.

[Insert Table II about here]

Similar to Table I, the *MSE* decreases, roughly proportionally, as the sample size increases. It drops by approximately two-thirds when the sample size increases from 50 to 150 observations.

II.C. *R*-square Decomposition

We derive an important extension of equation (3), dividing it by the estimated variance of asset j's returns ($\hat{\sigma}_j^2$). We are then able to decompose the estimate of *R*-square, i.e. the coefficient of determination, as follows:

$$R_j^2 = \sum_{k=1}^{K} DR_{j,k}^2, \text{ where } DR_{j,k}^2 = \left(\hat{\beta}_{k_j}^{\perp}\frac{\hat{\sigma}_{f^k}}{\hat{\sigma}_j}\right)^2. \tag{17}$$

Note that since the idiosyncratic risk can be measured as $\left(1-R^2\right)$, the sum of individual decomposed systematic risk measures and idiosyncratic risk equals one. In addition, from a statistical viewpoint, the decomposition of *R*-square characterizes the segments of goodness-of-fit. Parts of the total *R*-square can now be allocated unequivocally to each orthogonalized factor, indicating their relative contribution to the variation in the dependent variable (in our case, the return on the risky asset j).

III. Empirical Illustration

We apply the orthogonal transformation procedure described in Section II, to monthly returns on five well known equity pricing factors found in Kenneth R. French's Data Library: *RM*, *SMB*, *HML*, *Mom* and *Rev*. Historical observations suggest that market equity, book-to-market ratio, past short- or long-term returns may be proxies for exposures to various sources of systematic risk, not captured by the CAPM beta, and hence generating return premiums. Risk-based explanations for the return premiums to these factors might consider, for instance, that the returns on the *HML* and *SMB* portfolios seem to predict GDP growth, and thus they may be proxies for business cycle risk.[15]

Once the transformation is performed, the decomposed systematic risk and decomposed *R*-square are calculated and analyzed for style and industry portfolios, obtained from the same data library. Table III reports the distribution moments and the correlation matrix of factors' monthly returns over a sample period from January 1931 to December 2008 for both the original and the orthogonally-transformed data.

[Insert Table III about here]

As expected (see Panel A), sample variances are identical before and after the orthogonal transformation. Although the other distributional moments are different between non-orthogonalized and orthogonalized data, that does not affect the effectiveness of our decomposed systematic risk measures. Importantly, after transformation, the risk premiums for all factors will change. In our case, the mean returns of *RM*, *HML* and *Mom* increased, while for *SMB* and *Rev*, they decreased.

Note in Panel B of Table III that the original *RM*, *SMB*, *HML*, *Mom*, and *Rev* are more or less correlated to each other, while after the orthogonal transformation they become uncorrelated. Importantly, the orthogonal factors maintain a high resemblance to their original counterparts: the correlation coefficients between the original and the orthogonally-transformed returns are very high (i.e. 0.97, 0.96, 0.92, 0.97 and 0.91, respectively).

[15] See, for instance, Liew and Vassalou (2000).

Next, using the orthogonally-transformed data, we estimate the "orthogonal" beta coefficients, from the five-factor regression model for eight style portfolios, characterized by *size*, *value/growth*, *momentum*, and *contrarian*, respectively. Table IV presents the results for the equal-weighted portfolios.

[Insert Table IV about here]

Apparently, the absolute value of the "orthogonal" betas $\left(\hat{\beta}^{\perp}\right)$ is generally higher than that of the "non-orthogonal betas"$\left(\hat{\beta}\right)$.[16] For instance, the non-orthogonal contrarian beta $\left(\hat{\beta}_{\mathrm{Re}v}\right)$ of the small-cap and contrarian-sensitive portfolio (*Small/Low Rev*) is 0.42; but its orthogonal beta $\hat{\beta}^{\perp}_{\mathrm{Re}v}$ equals 0.95. Since the volatility estimates, $\hat{\sigma}$, are identical with or without orthogonalization, lower $\hat{\beta}$ (as compared to $\hat{\beta}^{\perp}$) indicates that the systematic risk is underestimated, if the correlation between factors is ignored.

The results presented in Tables III and IV highlight the importance of the orthogonal transformation in determining the proper risk premiums and beta coefficients, and suggest that multi-factor market models should be used cautiously. Paired *t*-tests indicate that, for the entire period (January 1931 to December 2008), the differences between the risk premiums on the orthogonal and the original factors were statistically significant for *Mom* and *Rev* (at 1 percent level), *HML* (at 5 percent level), *SMB* (at 10 percent level), and barely significant for *RM*.

We note that although the orthogonal beta $\left(\hat{\beta}^{\perp}\right)$ assesses the sensitivity of an asset's return to the variation in the underlying component of a factor's premium, it alone can not be used as a decomposed risk measurement. The appropriate approach is to take the product of $\hat{\beta}^{\perp}$-

[16] Again, the "orthogonal" betas are the coefficients obtained from the regression on the orthogonally-transformed factors, or simply by multiplying the correlation matrix (Ψ) by the original beta estimates. So, they are not orthogonal per se, but for ease of understanding, we prefer to term them accordingly.

square and the factor's volatility $\left(\text{i.e. } \hat{\beta}_{f_j}^{\perp^2} \hat{\sigma}_f^2\right)$. Table V shows the empirical results of systematic risk decomposition. Again, our methodology is applied to the eight style portfolios characterized by *size*, *value/growth*, *momentum*, and *contrarian*, respectively. We calculate the risk estimates for both the equal-weighted (Panel A) and value-weighted portfolios (panel B). We use an Ordinary Least Squares (OLS) regression model and we calculate the systematic variation ($\hat{\sigma}_{sr_j}^2$), taking the difference between the variance of a portfolio's returns ($\hat{\sigma}_j^2$) and its residual variance ($\hat{\sigma}_{\varepsilon_j}^2$) from the regression. It is clear, from both Panel A and Panel B, that the sum of decomposed systematic risk measures after the orthogonal transformation is exactly equal to the overall systematic variation ($\hat{\sigma}_{sr_j}^2$). This evidence shows the decomposability of risk measurement through orthogonalization.

Again, due to the correlations between the original factors, as reported in Table III, the sum of the non-orthogonal measures, $\hat{\beta}_{f_j}^2 \hat{\sigma}_f^2$, is clearly different from $\hat{\sigma}_{sr_j}^2$ (and generally much smaller).

[Insert Table V about here]

To examine the magnitude of individual decomposed risk measures in relation to the systematic and idiosyncratic risk, we further compute the decomposed *R*-squares by simply dividing $\hat{\beta}_{f_j}^{\perp^2} \hat{\sigma}_f^2$ by $\hat{\sigma}_j^2$. The sum of the decomposed *R*-squares stands for the overall systematic risk, and one minus the *R*-square becomes an assessment of the idiosyncratic risk. In addition, the decomposition of *R*-squares can be used to discriminate against unimportant factors in model specification. Table VI presents the risk decomposition for eighteen style portfolios (both equal-weighted and value-weighted): the six portfolios formed on *Size* and *Book-to-Market*, the six portfolios formed on *Size* and *Momentum*, and the six portfolios formed on *Size* and *Long-Term Reversal*.

[Insert Table VI about here]

The percentages of risk contributed systematically from *Market* (*RM*), *Size* (*SMB*), *Value* (*HML*), *Momentum* (*Mom*), and *Contrarian* (*Rev*) for the equal-weighted (value-weighted) small-cap portfolios are, on average, 53% (63%) , 25% (21%), 5% (5%), 6% (5%) and 5% (4%), respectively. This indicates that the systematic return-variation of the small-cap funds is, in a proportion of approximately 80 percent, caused by two sources: the overall *market* risk and the volatility of the *size* factor. The idiosyncratic risk of the value-weighted small-cap funds (around 2 percent) seems to be lower than that of the equal-weighted funds (approximately 5 percent).

For large-cap portfolios, on the other hand, roughly 80 percent of volatility comes from the *market* factor alone, while the *size* factor is relatively unimportant. Specifically, the decomposed *R*-squares of *size* are, on average, only 4 percent for the equal-weighted portfolios and 1 percent for the value-weighted funds. The unsystematic risk is about 3 to 4 percent.

In summary, from the point of view of modeling specification, the conventional single index market model is valid for large-cap stocks, but one needs to consider the *size* factor for small-cap stocks. Furthermore, although the *value*, *momentum* and *contrarian* factors seem to be unimportant for average portfolios of both large-cap and small-cap stocks, they do have some impact on the volatility of the style funds that carry their names. For instance, 16.43 (18.15) percent of the equal-weighted (value-weighted) large-cap/value fund's volatility comes from the value-factor (*HML*). The equal-weighted (value-weighted) large-cap/down-momentum fund has 23.43 (27.44) percent of the *R*-square contributed by the *momentum* factor (*Mom*). The decomposed *R*-squares of low-reversal portfolios with respect to the *Rev* factor range from 9.94% to 18.10%. This indicates that the factor-specification in market models is heterogeneous and varies by different styles of portfolio formation.

When comparing Tables IV and VI, it is even more interesting that higher original betas (in absolute value) of one factor versus another factor do not necessarily imply a relatively higher importance of the former. For example, the four small-cap portfolios in Table IV have higher betas of *SMB* compared to *RM*, still their corresponding decomposed-R^2 values are lower.

Next, our risk decomposition procedure is applied to monthly returns on 30-industry portfolios. As shown in Table VII, the unsystematic variation of industry funds is much larger than that of style portfolios. It ranges from 7.76% (*Fabricated Products and Machinery*) to 56.23% (*Tobacco Products*) for the equal-weighted portfolios and from 13.69% (*Banking, Insurance, Real Estate, Trading*) to 71.97% (*Coal*) for the value-weighted portfolios. The equal-weighted funds have larger DR^2_{SMB} than the value-weighted portfolios. This confirms again that the *size* factor is critical for pricing small-cap stocks. In addition, the *value*-factor has some weak influences on return variation of equal-weighted industry portfolios. For example, the decomposed *R*-square for the *value*-factor for *Transportation*, *Utilities*, *Finance*, and *Coal* are 12.12%, 9.00%, 8.68%, and 8.51%, respectively. It appears that the impact of the *momentum* and *contrarian* factors on the industry portfolios is small and relatively insignificant. Nevertheless, the high unsystematic risk $(1-R^2)$ suggests that other factors, specific to particular industries, may be influential.

[Insert Table VII about here]

From the overall sample analysis for the period ranging from January 1931 to December 2008, more than 85 percent, on average, of style portfolios' return variation is attributed to the DR^2 of *RM*, *SMB* and *HML*. This indicates that the Fama-French Three-Factor Model quantifies fairly well the risk-return structure of well-diversified equity portfolios. However, it is well known that the volatility of stock portfolios changes over time.[17] An examination of the time variation of equity risk decomposition is important. Monthly *R*-squares and decomposed *R*-squares are computed based on overlapping regression estimation for every 60-month (*t-59* to *t*) window, over a period ranging from January 1936 to December 2008. We illustrate, in Figure 1, the dynamic risk-decomposition for the value-weighted *Small / Value* and *Big / Value* style portfolios. In this case, for ease of comparability, they are selected from the Fama and French 25 portfolios formed on size and book-to-market. Again, the volatility is decomposed linearly into six components: *market* (DR^2_{RM}), *size* (DR^2_{SMB}), *value* (DR^2_{HML}), *momentum* (DR^2_{Mom}), *contrarian* (DR^2_{Rev}), and *idiosyncratic risk* ($1-R^2$).

[Insert Figure 1 about here]

In general, the largest component of return variation is captured by the *market* factor. *RM* maintains a similar importance when going from *Small* to *Big* (DR^2_{RM} for *Small* and for *Big* have a high correlation of 77 percent). Conversely, *SMB* is highly significant for *Small* and insignificant for *Big*, while *HML* is moderately more significant for *Big*. These results are in line with Fama and French (1993). Additionally, as expected, the idiosyncratic component is consistently higher for *Big*. In the two cases, both DR^2_{Mom} and DR^2_{Rev} have a low contribution: with a few exceptions they fall below 20 percent. From Figure 1, it appears that the decomposed components of risk not only are dynamic over time, but they may also exhibit significant correlations.[18] For example, DR^2_{Rev} (for *Small*) and DR^2_{HML} (for *Big*) move inversely with DR^2_{RM} (with correlation coefficients of -62 percent and -67 percent, respectively).

[17] For a good analysis of the dynamic nature of stock market volatility and idiosyncratic risk, see Campbell, Lettau, Malkiel, and Xu (2001).

[18] Finding the possible reasons for the ups and downs in the individual components over time is beyond the purpose of this paper, and we leave it to future research.

IV. Conclusions

Multi-factor models employing additional factors to the *market risk premium*, such as *size*, *value*, *momentum* and *contrarian*, have been widely used by financial researchers and professionals. Due to the dependence among factors, decomposing the systematic variation of asset returns, with respect to different factors has been a methodological challenge. This study aims to fill this gap and proposes a simple procedure of decomposing the coefficient of determination or *R*-square. This procedure allows us to examine the marginal contribution of individual factors to an asset's return volatility. The key component of our procedure is a simultaneously orthogonal transformation of data, that is able to extract jointly, the underlying uncorrelated components of individual factors. The covariance between the original factors is eliminated symmetrically, such that we achieve a maximum overall resemblance between the original and the transformed data sets. Experimentally, it appears that the decomposition is robust even for small sample sizes.

The decomposition procedure is further applied to return data on U.S. equity portfolios, obtained from *Kenneth French's Data Library*. Generally, the return variation of well-diversified equity portfolios is explained, with the highest proportion, by the *market risk premium* and *size* factors (in that order). Nevertheless, the decomposed elements of systematic risk and the systematic risk itself change over time.

As expected, the industry portfolios, both equal and value-weighted, exhibit significantly higher unsystematic risk than well-diversified portfolios. But, the decomposed-R^2 of the equally-weighted 30-industry portfolios favor *market* less, and the other factors more, compared to their value-weighted counterparts.

In summary, the paper provides a simple method to extract underlying (i.e. core) uncorrelated components from a set of correlated factors. This allows us to break the systematic variation of asset returns and observe the marginal contribution of risk from individual factors.

We note that the orthogonal transformation is numerical, and further research for developing a formal statistical process is necessary.

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Table I
The Mean Squared Errors (MSE) of the Decomposable Systematic Risk Estimates

This table presents the Monte Carlo simulation results (10,000 trials) for the *MSE* of our decomposed systematic risk estimates. We generate data for a five-factor linear model that has the following structure: $r = 0.02 + 1.2\, f_1 + 0.1\, f_2 + 0.3\, f_3 - 0.2\, f_4 + 0.7\, f_5 + e$, where f_k, $k = 1, 2,\ldots,5$, follow two hypothetical forms of distribution, multivariate-normal and multivariate-lognormal, respectively. The correlation coefficients of the original data range from approximately -0.4 to 0.7. The residuals, e, follow by turns, one of the two processes: homoscedastic [white noise ~ N(0, 0.21)] and heteroscedastic [GARCH(1,1), with coefficients 0.001, 0.5, 0.3].

$$MSE = E\left(\sum_{k=1}^{5} \left(\hat{\beta}_{k_j}^{\perp} \hat{\sigma}_{f^k}^{\perp} \right)^2 - \sigma_{sr}^2 \right)^2$$

Sample Size	Normal f_k, $k = 1, 2,\ldots,5$		Log-Normal f_k, $k = 1, 2,\ldots,5$	
	Homoscedastic (e)	Heteroscedastic (e)	Homoscedastic (e)	Heteroscedastic (e)
50	0.02373	0.02198	0.00119	0.00013
150	0.00817	0.00720	0.00038	0.00004
300	0.00404	0.00354	0.00019	0.00002
500	0.00232	0.00217	0.00011	0.00001
1000	0.00121	0.00110	0.00006	0.00001

Table II
Sampling Errors of Individual Decomposed Systematic Risk Estimates

This table presents the Monte Carlo simulation results (10,000 trials) for the *MSE* of the *individual* decomposed systematic risk estimates. We define a population with a finite number of observations (25,000), such that the generated data corresponds to the following five-factor linear model: $r = 0.02 + 1.2\, f_1 + 0.1\, f_2 + 0.3\, f_3 - 0.2\, f_4 + 0.7\, f_5 + e$, where f_k, $k = 1, 2,\ldots,5$, follow a multivariate-normal distribution. The correlation coefficients of the original data range from approximately -0.4 to 0.7. The distribution of the residuals (e) is normal, with a mean of zero and a standard deviation of 0.21.

$$MSE = E\left[\left(\hat{\beta}_k^{\perp}\hat{\sigma}_{f_k}^{\perp}\right)^2 - \left(\beta_k^{\perp}\sigma_{f_k}^{\perp}\right)^2\right]^2, \text{ for } k = 1, 2, \ldots, 5$$

Sample Size	f_1	f_2	f_3	f_4	f_5
50	0.01268	0.00035	0.00070	0.00090	0.00134
150	0.00398	0.00011	0.00022	0.00031	0.00043
300	0.00202	0.00005	0.00011	0.00015	0.00021
500	0.00123	0.00003	0.00006	0.00009	0.00012
1,000	0.00058	0.00001	0.00003	0.00005	0.00006

Table III
Distribution Properties of Factors' Sample Returns: Original vs. Orthogonal

This table reports the distribution parameters (Panel A) and correlation coefficients (Panel B) for the monthly returns on five stock-market factor portfolios: *RM*, *SMB*, *HML*, *Mom* and *Rev*, both non-orthogonalized and orthogonalized. *RM*, *SMB* and *HML* are the Fama/French factors: *RM* is the market risk premium; *SMB* is Small Minus Big, while *HML* is High Minus Low. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. All five factors follow the description and are obtained from Kenneth French's data library at http://mba.tuck.dartmouth.edu /pages/faculty/ken.french/. The sample period is January 1931 to December 2008. The orthogonalized measures, denoted by the symbol "$^{\perp}$", are obtained using the Schweinler – Wigner / Löwdin (1970) procedure.

Panel A: Distribution Parameters

	Original Returns					*Orthogonal Returns*				
	RM	*SMB*	*HML*	*Mom*	*Rev*	$RM^{\perp}$	$SMB^{\perp}$	$HML^{\perp}$	$Mom^{\perp}$	$Rev^{\perp}$
Mean	0.61	0.29	0.44	0.70	0.35	0.67	0.23	0.55	0.99	0.17
Std. Dev.	5.40	3.36	3.61	4.71	3.54	5.40	3.36	3.61	4.71	3.54
Skewness	0.30	2.29	1.91	-3.04	2.95	-0.24	1.44	0.63	-2.04	1.97
Kurtosis	8.34	22.99	16.11	28.33	24.22	4.82	14.89	5.54	15.45	17.40

Panel B: Correlation Coefficients

	Original Returns						*Orthogonal Returns*				
Factor	*RM*	*SMB*	*HML*	*Mom*	*Rev*	**Factor**	$RM^{\perp}$	$SMB^{\perp}$	$HML^{\perp}$	$Mom^{\perp}$	$Rev^{\perp}$
RM	1	0.33	0.23	-0.34	0.24	$RM^{\perp}$	1	0.00	0.00	0.00	0.00
SMB		1	0.10	-0.15	0.41	$SMB^{\perp}$		1	0.00	0.00	0.00
HML			1	-0.40	0.61	$HML^{\perp}$			1	0.00	0.00
Mom				1	-0.24	$Mom^{\perp}$				1	0.00
Rev					1	$Rev^{\perp}$					1

Table IV
Non-orthogonal vs. Orthogonal Beta Estimates

This table presents the estimated beta coefficients with and without an orthogonal transformation. We use monthly equal-weighted excess returns on eight style portfolios, obtained from Kenneth French's Data Library, for the time interval January 1931 – December 2008 (936 observations). By employing an OLS regression model, we estimate the beta coefficients with respect to five factors: *RM*, *SMB*, *HML*, *Mom* and *Rev*, respectively. *RM*, *SMB* and *HML* are the Fama/French factors: *RM* is the market risk premium; *SMB* is Small Minus Big, while *HML* is High Minus Low. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. The orthogonal measures are denoted by the symbol "$^{\perp}$". The values of *t*-statistics are reported in parentheses (one, two or three asterisks denote significance levels of 10%, 5% and 1%, respectively).

		Original Data (Correlated Returns)					**Orthogonally Transformed Data (Uncorrelated Components)**				
		$\hat{\beta}_{RM}$	$\hat{\beta}_{SMB}$	$\hat{\beta}_{HML}$	$\hat{\beta}_{Mom}$	$\hat{\beta}_{Rev}$	$\hat{\beta}^{\perp}_{RM}$	$\hat{\beta}^{\perp}_{SMB}$	$\hat{\beta}^{\perp}_{HML}$	$\hat{\beta}^{\perp}_{Mom}$	$\hat{\beta}^{\perp}_{Rev}$
Small	*Growth*	1.0574	1.2421	-0.0920	-0.1910	-0.0497	1.2014	1.3960	0.0570	-0.4066	0.3140
		(71.27)***	(49.58)***	(-3.35)***	(-11.04)***	(-1.74)*	(91.01)***	(65.74)***	(2.89)***	(-26.83)***	(15.60)***
	Value	0.9608	1.1961	0.7280	-0.1635	0.1746	1.1754	1.3774	0.8635	-0.5134	0.7587
		(72.90)***	(53.74)***	(29.81)***	(-10.64)***	(6.87)***	(100.22)***	(73.01)***	(49.23)***	(-38.13)***	(42.42)***
	Down Mom	0.9905	1.2869	0.3427	-0.5859	0.1473	1.2496	1.4907	0.5957	-0.8649	0.6730
		(54.10)***	(41.63)***	(10.10)***	(-27.43)***	(4.17)***	(76.70)***	(56.88)***	(24.45)***	(-46.25)***	(27.09)***
	Low Rev	0.9855	1.3605	0.4973	-0.2119	0.4186	1.2281	1.5944	0.7413	-0.5512	0.9482
		(53.96)***	(44.12)***	(14.69)***	(-9.95)***	(11.88)***	(75.57)***	(60.99)***	(30.50)***	(-29.55)***	(38.26)***
Big	*Growth*	1.0596	0.1975	-0.2226	-0.0915	-0.0556	1.0504	0.3913	-0.0842	-0.2433	0.0476
		(142.42)***	(15.72)***	(-16.15)***	(-10.54)***	(-3.87)***	(158.65)***	(36.74)***	(-8.51)***	(-32.01)***	(4.72)***
	Value	1.1188	0.2258	0.8116	-0.1189	-0.0012	1.1968	0.4413	0.8944	-0.4569	0.4438
		(115.98)***	(13.86)***	(45.40)***	(-10.56)***	(-0.07)	(139.43)***	(31.97)***	(69.68)***	(-46.37)***	(33.90)***
	Down Mom	1.1379	0.2065	0.1419	-0.6348	-0.0227	1.2410	0.4547	0.3867	-0.8455	0.2580
		(128.65)***	(13.82)***	(8.66)***	(-61.53)***	(-1.33)	(157.68)***	(35.92)***	(32.86)***	(-93.59)***	(21.49)***
	Low Rev	1.1225	0.1000	0.2132	-0.0957	0.6474	1.1856	0.4589	0.5447	-0.3778	0.8175
		(107.02)***	(5.65)***	(10.97)***	(-7.82)***	(31.98)***	(127.02)***	(30.57)***	(39.03)***	(-35.27)***	(57.44)***

Table V
Orthogonal Transformation and Systematic Risk Decomposition

This table presents decomposed systematic risk measures with and without an orthogonal transformation. We use monthly equal-weighted (Panel A) / value-weighted (Panel B) excess returns on eight style portfolios, obtained from Kenneth French's Data Library for the time interval January 1931 – December 2008 (936 observations) and we calculate the beta coefficients and factor variances with respect to five factors: *RM*, *SMB*, *HML*, *Mom* and *Rev*, respectively. *RM*, *SMB* and *HML* are the Fama/French factors: *RM* is the market risk premium; *SMB* is Small Minus Big, while *HML* is High Minus Low. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. The systematic variation σ_{sr}^2 is also computed, based on equations (1) and (2). In addition, after employing an orthogonal transformation of the aforementioned factors, we re-estimate the beta coefficients. The orthogonal measures are denoted by the symbol "$\perp$".

		Original Data (Correlated Returns)								Orthogonally Transformed Data (Uncorrelated Components)				
		$\hat{\beta}^2_{RM_j}\hat{\sigma}^2_{RM}$	$\hat{\beta}^2_{SMB_j}\hat{\sigma}^2_{SMB}$	$\hat{\beta}^2_{HML_j}\hat{\sigma}^2_{HML}$	$\hat{\beta}^2_{Mom_j}\hat{\sigma}^2_{Mom}$	$\hat{\beta}^2_{Rev_j}\hat{\sigma}^2_{Rev}$	Sum	$\hat{\sigma}^2_{sr_j}$	Sum	$\hat{\beta}^{\perp 2}_{RM_j}\hat{\sigma}^2_{RM}$	$\hat{\beta}^{\perp 2}_{SMB_j}\hat{\sigma}^2_{SMB}$	$\hat{\beta}^{\perp 2}_{HML_j}\hat{\sigma}^2_{HML}$	$\hat{\beta}^{\perp 2}_{Mom_j}\hat{\sigma}^2_{Mom}$	$\hat{\beta}^{\perp 2}_{Rev_j}\hat{\sigma}^2_{Rev}$
Panel A: Equal-Weighted Portfolios														
Small	Growth	32.635	17.408	0.111	0.808	0.031	50.993	69.060	69.060	42.131	21.987	0.042	3.663	1.237
	Value	26.948	16.140	6.919	0.592	0.383	50.982	84.527	84.527	40.326	21.404	9.732	5.839	7.226
	Down Mom	28.638	18.685	1.533	7.605	0.273	56.734	97.544	97.544	45.584	25.070	4.632	16.572	5.685
	Low Rev	28.352	20.884	3.228	0.995	2.199	55.659	97.899	97.899	44.028	28.681	7.174	6.732	11.285
Big	Growth	32.775	0.440	0.647	0.186	0.039	34.086	35.364	35.364	32.205	1.727	0.093	1.311	0.028
	Value	36.535	0.575	8.597	0.313	0.000	46.020	61.546	61.546	41.811	2.198	10.441	4.624	2.472
	Down Mom	37.795	0.481	0.263	8.930	0.006	47.475	65.914	65.914	44.956	2.333	1.952	15.837	0.835
	Low Rev	36.777	0.113	0.593	0.203	5.261	42.947	58.829	58.829	41.028	2.376	3.873	3.163	8.389
Panel B: Value-Weighted Portfolios														
Small	Growth	35.031	12.247	0.573	0.077	0.005	47.934	60.757	60.757	41.353	16.771	0.036	1.527	1.070
	Value	30.189	9.482	7.277	0.043	0.020	47.011	69.520	69.520	39.879	13.301	8.759	3.351	4.231
	Down Mom	33.885	10.951	0.820	6.991	0.003	52.649	84.143	84.143	47.599	15.982	2.940	14.854	2.768
	Low Rev	32.547	12.922	2.102	0.179	1.692	49.442	80.699	80.699	44.267	19.441	5.001	3.788	8.202
Big	Growth	29.899	0.119	0.732	0.010	0.001	30.760	27.709	27.709	26.914	0.124	0.123	0.538	0.011
	Value	34.718	0.006	8.500	0.028	0.001	43.253	52.456	52.456	37.316	0.621	9.800	2.887	1.832
	Down Mom	33.913	0.063	0.017	10.154	0.000	44.146	56.307	56.307	38.473	0.370	1.157	15.895	0.412
	Low Rev	33.221	0.237	0.026	0.046	8.436	41.966	49.210	49.210	34.934	0.691	2.384	1.907	9.294

Table VI
Risk Decomposition for Style Portfolios

We use three sets of monthly return data of style portfolios from Kenneth French's Data Library, including: the 6 Portfolios Formed on Size and Book-to-Market, the 6 Portfolios Formed on Size and Momentum, and the 6 Portfolios Formed on Size and Long-Term Reversal (all for the time interval January 1931 – December 2008). The R^2 value is calculated based on the five-factor model in equation (1), while $1-R^2$ is a measure of idiosyncratic risk. Further, by employing an orthogonal transformation of common risk factors and determining orthogonal beta coefficients, we calculate, according to equation (17), the decomposed-R^2 values with respect to each factor. The sum of all decomposed-R^2 and $1-R^2$ equals 100%. Two panels are reported in this table: (A) equal-weighted and (B) value-weighted portfolios, respectively.

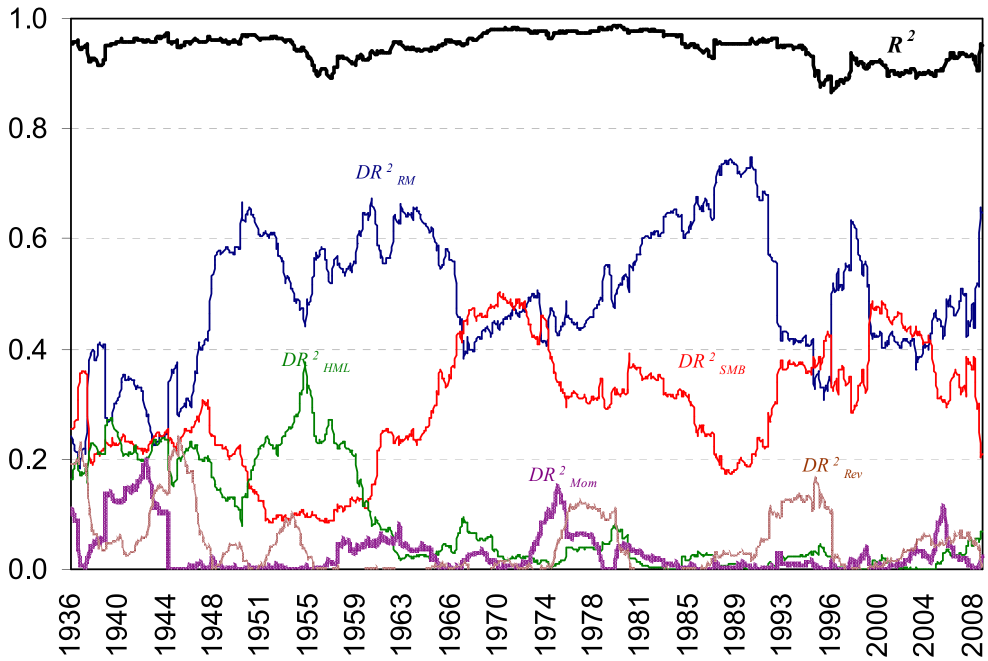
		Small							Big					
		Decomposed-R^2							**Decomposed-R^2**					
		RM	***SMB***	***HML***	***Mom***	***Rev***	$1-R^2$		***RM***	***SMB***	***HML***	***Mom***	***Rev***	$1-R^2$
Panel A: Equal-Weighted Portfolios														
BE/ME	Growth	57.09%	29.80%	0.06%	4.96%	1.68%	6.41%		88.10%	4.73%	0.25%	3.59%	0.08%	3.26%
	Neutral	58.09%	25.02%	3.82%	5.10%	4.58%	3.40%		80.12%	3.20%	5.54%	5.05%	1.94%	4.15%
	Value	45.69%	24.25%	11.03%	6.62%	8.19%	4.23%		65.80%	3.46%	16.43%	7.28%	3.89%	3.15%
Momentum	Down	43.52%	23.93%	4.42%	15.82%	5.43%	6.88%		66.51%	3.45%	2.89%	23.43%	1.24%	2.49%
	Medium	53.39%	23.26%	7.68%	5.34%	5.76%	4.58%		81.98%	3.05%	4.14%	6.04%	1.03%	3.76%
	Up	59.37%	27.14%	2.51%	0.01%	6.17%	4.79%		87.08%	6.79%	0.57%	1.68%	0.84%	3.03%
Reversal	Low	41.90%	27.30%	6.83%	6.41%	10.74%	6.82%		67.05%	3.88%	6.33%	5.17%	13.71%	3.86%
	Medium	57.99%	19.99%	7.48%	6.69%	4.68%	3.18%		80.18%	2.73%	5.87%	4.36%	2.77%	4.09%
	High	62.65%	21.41%	3.83%	7.51%	0.28%	4.32%		88.44%	4.03%	0.78%	3.38%	0.55%	2.82%
Panel B: Value-Weighted Portfolios														
BE/ME	Growth	66.52%	26.98%	0.06%	2.46%	1.72%	2.26%		95.10%	0.44%	0.43%	1.90%	0.04%	2.09%
	Neutral	66.86%	20.87%	3.99%	2.74%	3.41%	2.13%		83.00%	0.17%	6.04%	4.25%	1.97%	4.56%
	Value	56.91%	18.98%	12.50%	4.78%	6.04%	0.79%		69.11%	1.15%	18.15%	5.35%	3.39%	2.85%
Momentum	Down	55.48%	18.63%	3.43%	17.31%	3.23%	1.92%		66.42%	0.64%	2.00%	27.44%	0.71%	2.79%
	Medium	63.17%	19.02%	6.12%	4.96%	3.58%	3.15%		86.78%	0.19%	2.24%	6.53%	0.29%	3.98%
	Up	68.76%	23.23%	1.38%	0.29%	3.81%	2.54%		92.26%	1.32%	0.43%	2.42%	0.10%	3.46%
Reversal	Low	53.67%	23.57%	6.06%	4.59%	9.94%	2.16%		68.03%	1.35%	4.64%	3.71%	18.10%	4.17%
	Medium	64.45%	16.13%	6.89%	5.67%	3.86%	3.01%		85.23%	0.23%	3.62%	3.04%	2.69%	5.19%
	High	68.44%	19.24%	2.89%	6.25%	0.01%	3.18%		93.26%	0.84%	0.17%	1.77%	1.21%	2.76%

Table VII
Risk Decomposition for the 30-Industry Portfolios

We apply the R^2 decomposition approach to monthly returns on 30-industry portfolios obtained from Kenneth French's Data Library. The sample period is January 1931 to December 2008. The R^2 value is calculated based on the five-factor model in equation (1), while $1 - R^2$ is a measure of idiosyncratic risk. After an orthogonal transformation of common risk factors, we calculate decomposed R^2 with respect to each factor. The sum of all decomposed R^2 and $1 - R^2$ equals 100%. Two panels are reported in this table: (A) equal-weighted and (B) value-weighted portfolios, respectively.

	Panel A: Equal-Weighted Portfolios						Panel B: Value-Weighted Portfolios					
	Decomposed-R^2						Decomposed-R^2					
Industry	***RM***	***SMB***	***HML***	***Mom***	***Rev***	***1 − R²***	***RM***	***SMB***	***HML***	***Mom***	***Rev***	***1 − R²***
FOOD	58.46%	12.45%	5.78%	5.11%	3.61%	14.58%	67.56%	0.07%	1.22%	1.53%	0.10%	29.51%
BEER	36.86%	15.75%	3.21%	0.98%	2.42%	40.78%	44.65%	4.29%	1.63%	0.28%	1.07%	48.08%
SMOKE	33.76%	3.81%	1.36%	2.39%	2.46%	56.23%	32.88%	0.04%	0.69%	1.22%	0.25%	64.92%
GAMES	51.26%	15.62%	2.98%	5.61%	2.73%	21.78%	60.47%	5.19%	1.01%	4.04%	1.57%	27.72%
BOOKS	52.68%	18.54%	2.65%	4.58%	2.09%	19.47%	64.85%	5.31%	1.68%	1.91%	0.89%	25.36%
HSHLD	57.45%	19.65%	2.30%	3.71%	3.84%	13.05%	66.36%	0.22%	0.08%	0.91%	1.43%	30.99%
CLTHS	44.65%	22.34%	5.44%	6.77%	4.11%	16.69%	44.66%	9.48%	0.06%	1.73%	0.32%	43.75%
HLTH	57.69%	17.53%	0.61%	1.49%	1.14%	21.54%	64.67%	0.16%	0.28%	0.93%	0.22%	33.73%
CHEMS	69.16%	11.15%	3.19%	4.13%	1.70%	10.66%	75.38%	0.08%	0.52%	2.19%	0.58%	21.25%
TXTLS	47.83%	18.14%	4.32%	5.37%	2.98%	21.36%	55.98%	9.82%	4.13%	4.49%	1.88%	23.70%
CNSTR	55.62%	21.03%	4.42%	4.49%	3.54%	10.89%	77.22%	4.14%	1.28%	1.97%	0.67%	14.71%
STEEL	59.94%	10.85%	7.19%	6.47%	2.86%	12.69%	65.43%	3.00%	3.45%	4.22%	1.63%	22.27%
FABPR	64.77%	17.55%	3.21%	4.02%	2.69%	7.76%	75.75%	4.23%	1.37%	3.83%	0.92%	13.90%
ELCEQ	63.66%	15.84%	1.72%	3.22%	2.27%	13.28%	77.36%	0.50%	0.31%	2.74%	1.29%	17.80%
AUTOS	59.08%	14.86%	4.65%	5.81%	4.04%	11.55%	63.32%	0.65%	2.25%	5.26%	2.65%	25.88%
CARRY	52.99%	12.68%	7.21%	3.74%	1.92%	21.46%	60.95%	3.50%	4.83%	2.63%	0.62%	27.47%
MINES	35.57%	11.98%	4.84%	2.23%	1.79%	43.60%	40.44%	4.32%	2.05%	1.01%	0.12%	52.06%
COAL	26.62%	11.52%	8.51%	3.04%	0.75%	49.55%	22.13%	3.66%	1.90%	0.00%	0.34%	71.97%
OIL	49.36%	6.96%	5.54%	0.77%	0.49%	36.87%	59.66%	0.06%	4.41%	0.26%	0.02%	35.59%
UTIL	46.08%	2.35%	9.00%	4.54%	5.28%	32.75%	51.66%	0.01%	6.59%	3.05%	1.42%	37.27%
TELCM	55.72%	10.63%	0.84%	4.15%	0.78%	27.88%	55.38%	0.00%	0.07%	3.70%	0.43%	40.42%
SERVS	39.49%	24.60%	0.36%	1.68%	0.99%	32.88%	29.23%	4.53%	3.16%	0.14%	0.01%	62.93%
BUSEQ	59.54%	19.89%	0.47%	2.73%	1.68%	15.70%	70.49%	2.61%	4.56%	1.18%	0.41%	20.75%
PAPER	53.88%	18.38%	3.49%	4.93%	2.51%	16.82%	71.15%	0.49%	0.91%	2.48%	0.24%	24.72%
TRANS	45.72%	16.94%	12.12%	7.41%	3.98%	13.83%	63.30%	3.35%	8.74%	5.42%	0.54%	18.64%
WHLSL	40.20%	29.64%	1.55%	2.52%	3.97%	22.11%	58.90%	11.13%	0.61%	0.90%	1.11%	27.35%
RTAIL	56.43%	18.29%	1.44%	5.73%	2.53%	15.58%	69.42%	1.68%	0.01%	3.11%	0.32%	25.45%
MEALS	44.20%	18.99%	1.32%	2.09%	0.67%	32.73%	54.04%	5.93%	0.46%	1.24%	0.05%	38.29%
FIN	56.98%	11.13%	8.68%	7.24%	4.17%	11.79%	75.16%	0.61%	3.55%	6.13%	0.84%	13.69%
Other	49.34%	22.70%	1.31%	3.13%	3.14%	20.38%	65.77%	5.58%	0.13%	1.35%	0.83%	26.33%

Panel A: Decomposed R-square for the *Small* / *Value* Portfolio



Panel B: Decomposed R-square for the *Big* / *Value* Portfolio

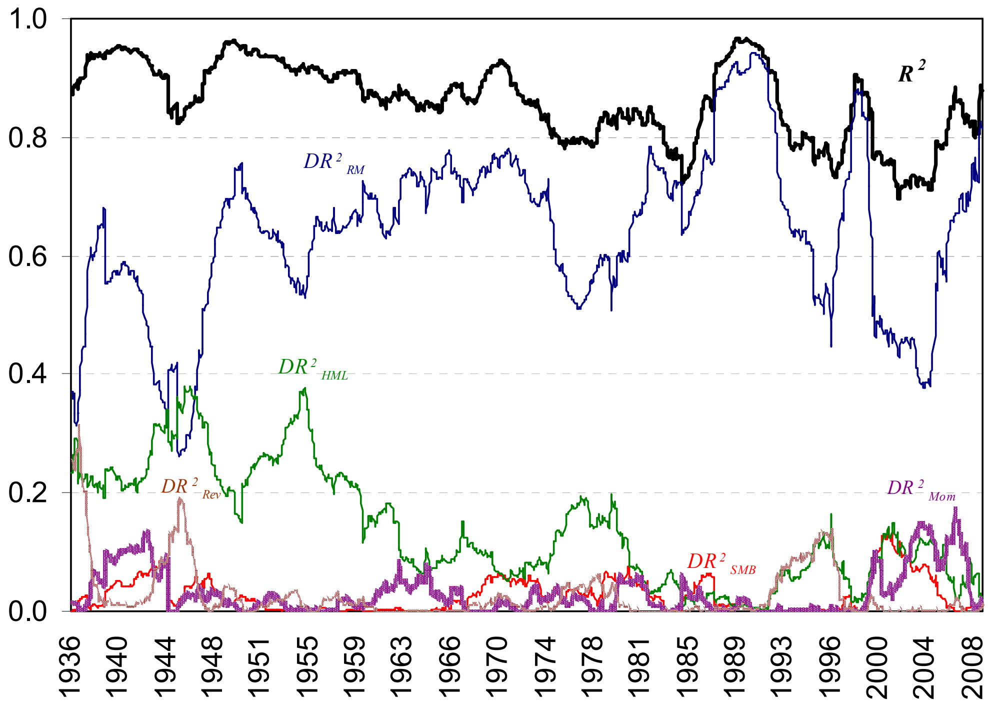


Figure 1. Decomposed Risk Over Time. The figure graphs the monthly variation in equity risk decomposition, from January 1936 to December 2008. The R^2 and the decomposed-R^2 (denoted by DR^2) are calculated based on overlapping regression estimation for every 60-month (*t-59* to *t*) window. Specifically, we present our empirical results on R^2 and DR^2 for two of the 25 value-weighted portfolios formed on Size and Book-to-Market (i.e. *Small* / *Value* and *Big* / *Value*), obtained from Kenneth French's data library at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/.

Appendix: The nature of the relationship between the original and the orthogonalized factors

Considering that each factor f^k can be written, according to equation (13) as

$f^k = \psi_{1k} f^{1^\perp} + \psi_{2k} f^{2^\perp} + \cdots + \psi_{Kk} f^{K^\perp}$, where $k = 1, 2, \ldots, K$ and the coefficients ψ_{lk} are the elements of the inverse of matrix $S_{K\times K}$, as transformed in equation (11), we want to prove that for any k and l, $\psi_{kl} = \operatorname{corr}\left(f^k, f^{l^\perp}\right)$.

Proof:

The inverse of matrix $S_{K\times K}$ (in its final form), can be written as follows:

$$S_{K\times K}^{-1} = \frac{1}{\sqrt{T-1}} \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sigma_K \end{bmatrix} \begin{bmatrix} \dot{s}_{11} & \dot{s}_{12} & \cdots & \dot{s}_{1K} \\ \dot{s}_{21} & \dot{s}_{22} & \cdots & \dot{s}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{s}_{K1} & \dot{s}_{K2} & \cdots & \dot{s}_{KK} \end{bmatrix} = \frac{1}{\sqrt{T-1}} \begin{bmatrix} \dot{s}_{11}/\sigma_1 & \dot{s}_{12}/\sigma_1 & \cdots & \dot{s}_{1K}/\sigma_1 \\ \dot{s}_{21}/\sigma_2 & \dot{s}_{22}/\sigma_2 & \cdots & \dot{s}_{2K}/\sigma_2 \\ \vdots & \vdots & \ddots & \vdots \\ \dot{s}_{K1}/\sigma_K & \dot{s}_{K2}/\sigma_K & \cdots & \dot{s}_{KK}/\sigma_K \end{bmatrix},$$

where $\dot{s}_{kl} = \dot{s}_{lk}$ (the inverse of a symmetric matrix is symmetric).

So, $\psi_{lk} = \frac{1}{\sqrt{T-1}} \dot{s}_{lk}/\sigma_l$, where k and l go from 1 to K.

Thus, $f^k = \frac{1}{\sqrt{T-1}} \left(\dot{s}_{1k}/\sigma_1 \, f^{1^\perp} + \dot{s}_{2k}/\sigma_2 \, f^{2^\perp} + \cdots + \dot{s}_{lk}/\sigma_l \, f^{l^\perp} + \cdots + \dot{s}_{Kk}/\sigma_K \, f^{K^\perp} \right)$.

We can now calculate the covariance between f^k and $f^{l^\perp}$ [Note that for any $i = 1, 2, \ldots, K$, $\operatorname{var}\left(f^{i^\perp}\right) = \operatorname{var}\left(f^i\right)$]:

$$\operatorname{cov}\left(f^k, f^{l^\perp}\right) = \frac{1}{\sqrt{T-1}} \dot{s}_{lk}/\sigma_l \operatorname{var}\left(f^{l^\perp}\right) = \frac{1}{\sqrt{T-1}} \times \dot{s}_{lk} \times \sigma_l$$

Similarly, $\operatorname{cov}\left(f^l, f^{k^\perp}\right) = \frac{1}{\sqrt{T-1}} \times \dot{s}_{kl} \times \sigma_k$.

But, $\dot{s}_{lk} = \dot{s}_{kl} \Rightarrow \dfrac{\operatorname{cov}\left(f^k, f^{l^\perp}\right)}{\sigma_l} = \dfrac{\operatorname{cov}\left(f^l, f^{k^\perp}\right)}{\sigma_k}$. (*)

Writing equation (13) for factor l, we have:

$f^l = \psi_{1l} f^{1^\perp} + \psi_{2l} f^{2^\perp} + \cdots + \psi_{kl} f^{k^\perp} + \cdots + \psi_{Kl} f^{K^\perp}$.

Thus, $\operatorname{cov}\left(f^{l}, f^{k^{\perp}}\right)=\psi_{kl} \times \operatorname{var}\left(f^{k^{\perp}}\right) \Rightarrow$

$$\psi_{kl}=\frac{\operatorname{cov}\left(f^{l}, f^{k^{\perp}}\right)}{\operatorname{var}\left(f^{k^{\perp}}\right)} \overset{\text{from }(*)}{=} \frac{\operatorname{cov}\left(f^{k}, f^{l^{\perp}}\right) \times \sigma_{k}}{\sigma_{l} \times \sigma_{k}^{2}}=\frac{\operatorname{cov}\left(f^{k}, f^{l^{\perp}}\right)}{\sigma_{l} \times \sigma_{k}}=\operatorname{corr}\left(f^{k}, f^{l^{\perp}}\right).$$

Q.E.D.