Effects of Decision Interval on Optimal Intertemporal Portfolios

With Serially Correlated Returns

1. Introduction

A challenging topic in the theory of portfolio choice is that of intertemporal portfolios in discrete time. In the absence of transactions costs—costs of collecting and processing information and implementing new portfolio decisions—the interval between decisions would be infinitesimal and so portfolio choice would occur in continuous time. But casual observation shows that for many decision makers such costs do exist, because we observe many people revisiting their portfolio allocations on a monthly, quarterly, or annual basis rather than each minute or day.

Hence arise these questions: How is the size of the risky portion of the portfolio influenced by the decision interval—the length of time over which current decisions will be in force, before new decisions based on updated information can be made and implemented? And how does the risky portion evolve within the decision interval if precommitted portfolio revisions are possible? Several papers have addressed these questions. In Goldman's (1979) model a longer holding period leads to portfolios which are less diversified between the risky and riskfree assets in the absence of serial correlation of returns, and Benninga and Blume (1988) found similar results. But Fischer and Pennacchi (1985) found that the unambiguous effect on diversification is lost if there is serial correlation of the risky asset return.\(^1\)

Samuelson (1991) considered a case of a two-state Markov process in which the portfolio share is reset to a precommitted constant value at the start of each time period within the decision interval. He found that with log utility and either no serial correlation or negative serial correlation, the length of the decision interval has no effect on the risky share; he also found that with a more risk
averse CRRA utility function, the decision interval still has no effect in the absence of serial correlation, but with negative serial correlation a longer decision interval leads to a higher value for the optimal risky share. Samuelson speculated that this result would be reversed by assuming either positive serial correlation or a utility function less risk averse than log utility. Finally, Balvers and Mitchell (2000) considered precommitted paths of the risky asset holding when a not necessarily constant sequence of risky holdings is precommitted to at the decision time. They found that under negative serial correlation, and possibly under positive serial correlation as well, the precommitted risky holding can be lower the farther into the future the precommitment is for.

Allowing for the possibility of serial correlation of risky asset returns, as do Fischer and Pennacchi (1985), Samuelson (1991), and Balvers and Mitchell (2000), can substantially complicate the theoretical analysis of portfolios. But serial correlation is an important consideration due especially to recent empirical work. Poterba and Summers (1988) and Fama and French (1988) found negative autocorrelation in annual stock returns, although Kim et al. (1991) suggested that this result is mostly attributable to the inclusion of pre-World War II data. Lo and MacKinlay (1988), on the other hand, found that weekly returns exhibit positive serial correlation. Thus it is important to know whether and how the presence of positive or negative serial correlation influences the effect of the decision interval on the risky asset share.

The purpose of this paper is to explore further the possible effects of the decision interval upon the risky asset's share in the portfolio, when there may be positive or negative serial correlation of risky returns. We consider both the case in which rebalancing is constrained to maintain a constant risky share at the start of each time period within the decision interval (as in Samuelson (1991)) and the case
in which the risky share must follow a precommitted but not necessarily constant sequence (as in Balvers and Mitchell (2000)).

For this purpose the log utility function is employed. There are several reasons that this choice of utility function is useful. It maintains a common literature custom of assuming constant relative risk aversion, and as is well-known it often serves as a qualitative borderline case between utilities with relative risk aversion greater than unity and those with relative risk aversion less than unity. In addition, it allows comparison of the results of this paper with Samuelson's (1991) above-mentioned result showing a context in which the decision interval has no effect under log utility. As it turns out, we find that Samuelson's result is not robust, and the convenient additive separability property of log utility allows us to develop some straightforward intuition for why it is not. Instead, we find that under arguably plausible and broad assumptions, a longer decision interval leads to a lower risky share if that share is constrained to be held at a constant level each period within the decision interval, and leads to a declining risky share if that share must be precommitted to but need not be constant. These results are independent of the sign of serial correlation, and can arise even in the absence of serial correlation, contrary to Samuelson's conjecture.

Section 2 very briefly presents the portfolio optimization problem and points out the convenient nature of log utility for our purposes. Section 3 postulates an ARMA \((p,q)\) process for the excess return on the risky asset, with arbitrary \(p\) and \(q\) and arbitrary (unspecified) stochastic distribution function for the innovations; with either positive or negative serial correlation it is shown that a longer decision interval induces a more conservative portfolio. Then, armed with the intuition deriving from Section 3, Section 4 replaces the ARMA excess returns process with a three-state Markov process, in
the spirit of Samuelson's two-state Markov process. Samuelson's result that, regardless of serial correlation, the decision interval has no effect on portfolio conservatism under log utility is shown not to hold up; and intuition is provided as to why not and as to how that result occurred in Samuelson's context. As with the ARMA process we find, under a plausible parameter restriction, that in the presence of positive, negative, or now even no serial correlation a longer decision interval induces a more conservative portfolio. Section 5 concludes the discussion.

2. Log Utility

In this paper we assume that utility is given by $U(\tilde{W}) = \ln \tilde{W}$ where $\tilde{W}$ is final wealth (that is, wealth at the end of the decision period). If in fact there are multiple sequential decision intervals, the well-known myopia property of log utility says that even with serial correlation of risky returns, the optimal portfolio sequence for the first decision interval is the same as that found by assuming there is only one decision interval. Thus it is harmless to use the term "final wealth" to refer to wealth at the end of the first decision interval.

If the decision interval consists of $T$ discrete periods of time, where the time periods are determined by the dynamic process generating risky returns, then final wealth $\tilde{W}$ is given by

$$\tilde{W} = W_0 [ R_f + w_1 \tilde{x}_1 ] \ldots [ R_f + w_T \tilde{x}_T ]$$

(1)

where initial wealth $W_0$ will henceforth be normalized to unity, $R_f$ is the riskfree return (one plus the riskfree rate), $w_t$ ( $t=1,...,T$ ) is the risky asset share for period $t$ precommitted at the start of the decision interval, and $\tilde{x}_t$ is the excess return on the risky asset (that is, in excess of $R_f$) in period $t$. With $W_0 = 1$ and using the additive separability of the log function, expected utility is given by
\[ E_0 U(\tilde{W}) = E_0 \ln [ R_f + w_1 \tilde{x}_1 ] + ... + E_0 \ln [ R_f + w_T \tilde{x}_T ] . \] (2)

The additive nature of Equation 2 allows each term to be optimized separately if there is no constant-share constraint. So with precommitment to an unconstrained risky share path, the problem of choosing \( w_t (t = 1, ..., T) \) is

\[ \max_{w_t} E_0 \ln [ R_f + w_i \tilde{x}_i ] . \] (3)

3. Results for an ARMA(\(p,q\)) Process

In this section we assume that the stochastic evolution of the excess return on the risky asset is given by an ARMA(\(p,q\)) process:

\[ \tilde{x}_t = (1 - \sum_{i=1}^{p} \tilde{\alpha}_i) b + \sum_{i=1}^{p} \tilde{\alpha}_i \tilde{x}_{t-i} - \sum_{j=1}^{q} \tilde{\beta}_j \tilde{u}_{t-j} + \tilde{u}_t . \] (4)

Here \( b > 0 \) is the unconditional (long-run) mean of \( \tilde{x} \). The innovation \( \{ \tilde{u}_t \} \) is independently and identically distributed according to any zero-mean distribution whose support satisfies the restrictions implying that the excess return \( x_t \) always has a positive mean and always has a non-zero probability of being negative.³

Clearly the time-zero expectation of the risky return path will influence the relative risk-taking of constant-share portfolios chosen for alternative decision intervals. For instance, suppose the initial conditions on the dynamic process are such that the risky excess return is expected to decline over time. This would naturally impart a tendency for portfolios with risky share locked in over a longer interval to hold less of the risky asset. This effect is obvious and is not what we seek to analyze here; so we assume neutral initial conditions such that the expected path of the risky excess return is constant. Hence the \( p \) excess returns prior to the decision time are assumed to equal the long-run mean value \( b \),
and the $q$ prior innovations are assumed to equal zero.

As is well-known, the ARMA process 4 can be written as an infinite-order MA process:

$$\tilde{x}_t = b + \tilde{u}_t + \sum_{i=1}^{4} c_i \tilde{u}_{t-i} .$$

(5)

Given $u_t = 0$ for all $t$ prior to the decision time at the start of period 1, Equation 5 becomes

$$\tilde{x}_t = b + \tilde{u}_t + \sum_{i=1}^{t-1} c_i \tilde{u}_{t-i} .$$

(6)

Thus by Equations 3 and 6 the problem of choosing each $w_t$ ($t=1,...,T$) at the start of the decision interval, with no constant-shares constraint, is

$$\max_{w_t} E_{0} \ln [ R_f + w_t ( b + \tilde{u}_t + \sum_{i=1}^{t-1} c_i \tilde{u}_{t-i} ) ] .$$

(7)

**Proposition 1:** _For an ARMA(p,q) excess returns process and log utility, if the risky shares $w_1$, ..., $w_T$ must be precommitted at the start of period 1, then with positive or negative serial correlation (that is, with $p \tilde{O} 0$ and/or $q \tilde{O} 0$) we have $w_1^* \leq w_2^* \leq ... \leq w_T^*$; and an inequality is strict with $w_s^* > w_{s+1}^*$ if and only if $c_s \tilde{O} 0$. Also, $w_t^*$ is independent of $T$ for all $t$. 

**Proof:** If the $\{\tilde{x}_t\}$ were serially independent, we would have $c_i = 0$ for all $i$, and since $u_t$ is identically distributed through time Problem 7 would be identical for all $t$, thus giving identical values of $w_t$ for all $t$. Therefore under serial independence we would obtain constancy of the risky share even without a constant-risky-share constraint.

But with serial correlation we have $c_i \tilde{O} 0$ for some values of $i$, so that the excess return expression appearing in Problem 7 has more terms for periods later in the decision interval. In fact, we can see that the excess return for $t=\delta$ is a mean-preserving spread of that for $t=s$ if $\delta > s$, as follows.

By Equation 6,
\[
\tilde{x}_{s+1} = b + \tilde{u}_{s+1} + \sum_{i=1}^{s} c_i \tilde{u}_{s+1-i} = b + \tilde{u}_{s+1} + \sum_{i=1}^{s-1} c_i \tilde{u}_{s+1-i} + c_s \tilde{u}_1.
\]  
(8)

The portion of this expression excluding the term \( c_s \tilde{u}_1 \) is equal in distribution to \( \tilde{x}_s \), since \( \{\tilde{u}_i\} \) is i.i.d.:

\[
\tilde{Y} / b + \tilde{u}_{s+1} + \sum_{i=1}^{s-1} c_i \tilde{u}_{s+1-i} \overset{d}{=} b + \tilde{u}_s + \sum_{i=1}^{s-1} c_i \tilde{u}_{s-i} = \tilde{x}_s.
\]  
(9)

Then

\[
\tilde{x}_{s+1} / \tilde{Y} + c_s \tilde{u}_1 \overset{MPS}{\overset{d}{=}} \tilde{x}_s
\]  
(10)

if \( c_s \tilde{O} 0 \). Here \( \tilde{Y} + c_s \tilde{u}_1 \) is a mean-preserving spread of \( \tilde{Y} \) because \( \tilde{u}_1 \) has zero mean and is independent of \( \tilde{Y} \). Hence we have that \( \tilde{x}_{s+1} \) is a mean-preserving spread of \( \tilde{x}_s \) if \( c_s \tilde{O} 0 \).

So by induction the problem for period \( \hat{o} \ (\hat{o} > s) \) differs from that of period \( s \) in that the risky return distribution in period \( \hat{o} \) is a mean-preserving spread of that in period \( s \) (if \( c_i \tilde{O} 0 \) for at least one of \( i = s, ..., \hat{o}-1 \)). Appendix 1 shows that log utility chooses to hold less of the risky asset when it has undergone this mean-preserving spread. QED.

Next consider Samuelson's problem of precommitting to a risky asset share that must stay constant for all periods \( t = 1, ..., T \) in the decision interval. Continue to denote as \( w_t^* \) the optimal share for period \( t \) in the absence of the fixed-share constraint, and denote as \( \hat{\omega}_T \) the optimal fixed share when the decision interval length is \( T \). Then by Proposition 1, under serial correlation we know \( w_1^* \lessgtr w_2^* \). With \( T = 2 \) it is intuitive that \( w_1^* \lessgtr \hat{\omega}_2 \lessgtr w_2^* \). Likewise, we know \( w_2^* \lessgtr w_3^* \), so it is intuitive that the addition of a third period over which the share \( \hat{\omega} \) must remain fixed would pull \( \hat{\omega} \) down: \( \hat{\omega}_2 \lessgtr \hat{\omega}_3 \lessgtr w_3^* \). This intuition is proven and inductively extended to all \( T \) in Appendix 2.

Therefore we have this result for the fixed-shares problem:

**Proposition 2:** For an ARMA(p,q) excess returns process and log utility, if there must be
a precommitment at the start of period one for a constant share $\phi_T$ to be implemented at the
start of each period $t=1,...,T$ during the decision interval of length $T$, where alternately
$T=2,...,K$, then $w_1^* > \phi_2 > \phi_3 > ... > \phi_K > w_K^*$. We have strict inequalities $w_1^* > \phi_2$ if
c_i \bar{O} 0, $\phi_s > \phi_{s+1}$ if $c_s \bar{O} 0$ ($s=2,...,K-1$), and $\phi_K > w_K^*$ if $c_i \bar{O} 0$ for any of $i=1,...,K-1$.

**Proof:** See Appendix 2.

This section has found that precommitting to decisions that extend farther into the future leads to
greater conservatism in the holding of the risky asset, in the presence of positive or negative serial
correlation in any ARMA process. The intuition of this basic result is made clear by the observation
that risky returns farther in the future are mean-preserving spreads of risky returns in the nearer future.
This makes sense: intuition strongly suggests that the nearer future is less of a mystery than the more
distant future, since more surprises can occur between now and a more distant future time. The
spread-out nature of future returns occurs under an ARMA process regardless of whether the serial
correlation is positive or negative, just so long as it is non-zero.

The strongly intuitive nature of this way of comparing nearer and farther future returns suggests
that our basic result should hold up under any reasonable process that might be assumed for the
evolution of the excess return, and not just for an ARMA process. Indeed, this straightforward
generalization follows immediately from the analysis of this section:

**Proposition 3:** Under log utility the inequality chains from Propositions 1 and 2 in their
strict form, $w_1^* > w_2^* > ... > w_T^*$ and $w_1^* > \phi_2 > \phi_3 > ... > \phi_K > w_K^*$, hold under any
dynamic process for the excess return on the risky asset, if that process has the feature that any
more distant excess return distribution is a mean-preserving spread of a nearer one.
Proof: Appendices 1 and 2, previously used in the proofs of Propositions 1 and 2, prove Proposition 3 as well.

It is reasonable to anticipate that, as with the ARMA process, any dynamic excess return process with either positive or negative serial correlation might have this mean-preserving spread property. But we are now faced with this question: why did the model of Samuelson (1991) give the conclusion that with log utility and negative serial correlation, the length of the decision interval over which a constant risky share must be precommitted has no effect on the risky share chosen? The focus of the next section is on answering this question and on seeing what happens when Samuelson's approach is generalized.

4. Results for a Finite-State Markov Process

Samuelson (1991) assumed that the dynamic excess returns process is a two-state Markov process. He considered both (a) one-period decision intervals with sequential decisions, and (b) a single multi-period decision interval with a constraint that a precommitted risky asset share be re-established at a fixed constant value at the start of each period within the decision interval. The utility function was assumed alternately to be \( \log W \) or \( -W^{-1} \).

Of interest for the present paper is the log utility function with a multi-period decision interval. In this context Samuelson found [pp. 188-191] that the optimal risky asset share, constrained to be constant across periods during the decision interval, is independent of the length of the decision interval, with or without serial correlation. This of course goes against the intuition in the present paper, based on the notion that the farther future is less knowable than the nearer future. But we shall see, by
examining Samuelson's set-up, that his specific framework had the effect of ruling out exactly this intuitive feature: in Samuelson's analysis of the effect of decision interval length, the future is implicitly assumed to be exactly as knowable as the present.

Samuelson assumed a two-state Markov process, whereby the risky excess return equals either \( x^+ \) or \( x^- \). His rebound probability--\( Pr( \tilde{x}_t = x^+ | x_{t-1} = x^- ) \) which also equals \( Pr( \tilde{x}_t = x^- | x_{t-1} = x^+ ) \)-- is 2/3. Thus the dynamic evolution of the vector \( \delta \) of state probabilities is given by

\[
\begin{bmatrix}
    1/3 & 2/3 \\
    2/3 & 1/3 
\end{bmatrix}
\begin{bmatrix}
    \delta^- \\
    \delta^+
\end{bmatrix} =
\begin{bmatrix}
    2/3 & 1/3 \\
    1/3 & 2/3 
\end{bmatrix}
\begin{bmatrix}
    \delta^-_{t-1} \\
    \delta^+_{t-1}
\end{bmatrix}.
\]

(11)

Recall that early in Section 3 we needed to assume neutral (long-run mean) initial conditions for lagged values \( x_{t-i} \) and \( u_{t-i} \) \( (i=1,2,\ldots) \) prior to the decision time, to avoid imparting a slope to the time path of future expected risky returns, since any such slope would create a trivially obvious tendency for the more distant future to be more or less attractive. Samuelson too needed to impose neutral initial conditions; but with only two states in his model--one above the long-run mean excess return and one below the long-run mean--it is impossible to set the lagged excess return equal to the long-run mean. So what Samuelson did was to assume rather artificially that at the decision time the lagged excess return is not known, but might be high or low with equal probabilities. This approach succeeds in creating a flat time path for future expected excess returns, but it imposes an unreasonable restriction on information availability--namely, that no lagged information is available. The assumption is that

\[
\begin{bmatrix}
    \delta^-_0 \\
    \delta^+_0
\end{bmatrix} = \begin{bmatrix}
    1/2 & 1/2
\end{bmatrix}; \text{ in fact this state probability vector is equal to the steady-state probability}
\]
vector of the Markov process 11. Thus as we iterate Equation 11 forward from the initial probability vector, the probability vector never changes. That is, by construction in Samuelson's model, the probability vector for any future excess return is identical to that for the first-period excess return!

So this set-up assumes away our intuition concerning the farther future being less knowable than the nearer future. As a result, the single-period optimization Problem 3 at the decision time for any future precommitted period \( t \) (in the case of no constant-share constraint) is identical for all \( t \), and hence the problem with a constant-share constraint yields the same solution regardless of how many periods the constant share is being precommitted for. This explains Samuelson's result, contrary to the implication of our Proposition 3, that the length of the decision interval does not matter: part of the premise of Proposition 3 is violated.

It is worthwhile to investigate the question of whether Samuelson's basic approach of using a finite-state Markov process can be retained, while imposing neutral initial conditions yet allowing the intuitive feature that the farther future is less knowable than the nearer future. There appears to be no alternative to Samuelson's way of imposing neutral initial conditions if there are only two states. But with three states, we can do so by imposing that one of the possible excess return values equals the long-run mean excess return, and simply setting the lagged excess return to this value. By assuming that this is the lagged value with certainty, we avoid the unrealistic informational assumption that Samuelson used (that the lagged excess return is unknown).

Suppose the excess return can take on any of three values, \( x^L \), \( x^M \), and \( x^H \), with \( x^M \) being the midpoint between \( x^L \) and \( x^H \), and suppose the transition dynamics for the state probability vector are given by
where in this transposed Markov matrix element $i,j$ is the contingent probability of going from state $j$ to state $i$ ($i,j = L, M, H$). The column sums are necessarily unity, and dynamic symmetry has been assumed in that $Pr( x_t^* = x_L \ | \ x_{t-1}^* = x_L) = a = Pr( x_t^* = x_H \ | \ x_{t-1}^* = x_H) = b = Pr( x_t^* = x_H \ | \ x_{t-1}^* = x_L)$, and $Pr( x_t^* = x_L \ | \ x_{t-1}^* = x_M) = c = Pr( x_t^* = x_H \ | \ x_{t-1}^* = x_M)$. We assume $1 - a - b > 0$, $1 - 2c > 0$, and $a, b, c > 0$. Then $x_0 = x^M$ is a neutral initial condition because $E_0( x_t^* = x_{t+1}^* = x^M \ | \ x_0^* = x_M)$ is preserved by the use of the neutral initial condition. Since we have just three states and distributional symmetry via $\tilde{\rho}_t = \rho_t$, a mean-preserving spread occurs from $t$ to $t+1$ if and only if $\tilde{\rho}_{t+1} > \rho_t$ (so probability mass has been moved from the center ($x^M$) to the lower and upper ends of the distribution).

To determine when this occurs, extract from Equation 12 the dynamics of $\tilde{\rho}_t$: (noting again that $\tilde{\rho}_t = \rho_{t-1}$):

$$\tilde{\rho}_t - \rho_t = (a+b-2c)[ \tilde{\rho}_{t-1} - \rho_t ]$$

where $\rho_t = c / (1-a-b+2c) > 0$ is the steady-state value of $\tilde{\rho}_t$ and $\rho_t$ -- that is,
\[
\lim_{t \to \infty} \Pr_0(\tilde{x}_t = x^L) = \delta^* = \lim_{t \to \infty} \Pr_0(\tilde{x}_t = x^H).
\]

The solution of difference equation 13 is

\[
\delta^{H}_{L} - \delta^{*} = (a+b-2c) \left[ \delta^{H}_{0} - \delta^{*} \right].
\]  

(14)

The eigenvalue \((a+b-2c)\) is in the range \((-1, 1)\).

**Proposition 4:** For the three-state dynamic excess returns process (12) with \(x^M = (x^L + x^H)/2\) and with the neutral initial condition \(x_0 = x^M\), and under parameter condition \(a + b - 2c > 0\), the following hold for log utility:

(a) If the risky shares \(w_1, ..., w_T\) must be precommitted at the start of period one, we have \(w_1^* > w_2^* > ... > w_T^*\), with \(w_t^*\) independent of \(T\) for all \(t\); and

(b) If there must be a precommitment at the start of period one for a constant share \(\tilde{\delta}_T\) to be implemented at the start of each period \(t = 1, ..., T\) during the decision interval of length \(T\), where alternately \(T = 2, ..., K\), then \(w_1^* > \tilde{\delta}_2 > \tilde{\delta}_3 > ... > \tilde{\delta}_K > w_K^*\).

**Proof:** By Equation 14, we have monotonic convergence of \(\delta^L\) (and of \(\delta^H\)) to \(\delta^*\) if and only if \(a + b - 2c > 0\). Now since our neutral initial condition has \(x_0 = x^M\) with certainty, \(\delta^{L}_0 = 0 = \delta^{H}_0\) and so under this monotonicity condition \(\delta^{L}_1\) and \(\delta^{H}_1\) rise each period, creating a further mean-preserving spread for each period farther into the future we look. Thus from Proposition 3 we have the indicated results for the symmetric three-state Markov process. QED.

Notice that again, as with the ARMA process, the positive or negative nature of the serial correlation is irrelevant to the effect of the decision interval on the demand for the risky asset: one can show that first-order serial correlation is positive, zero, or negative as \((a-b)\) is positive, zero, or negative; but the monotonicity condition that \(a + b - 2c\) be positive is unaffected by the sign of \((a -
b). In fact, even with $a = b$, which gives zero serial correlation at all lags, the monotonicity condition is still met if $a > c$. In this case we have sequential mean-preserving spreads and hence the inequality chains of Proposition 4 even with no serial correlation, a possibility that by Proposition 1 does not exist under the ARMA $(p,q)$ structure for excess returns. This shows that in general, not only is the sign of serial correlation not relevant for determining the portfolio effect of decision interval length, but the very presence or absence of serial correlation is not the determining factor.

It remains to interpret this condition for monotonic spreading over time. The condition requires $a + b > 2c$, which is to say that an outlier (equaling either $x^L$ or $x^H$) in one period is more likely than a central value (equaling $x^M$) to lead to an outlier (the same one or the opposite one) in the next period. It is indeed intuitive that this condition would avoid oscillations, and it is natural to impose it in light of our intuition that the more distant future is always more uncertain than the less distant future.

5. Summary and Literature Comparison

This paper has considered situations in which the presence of transactions costs leads to decision intervals of multiple periods in length, so that decisions on the risky portfolio share sequence must be precommitted to at the start of the decision interval; the share either does or does not have to remain constant across time periods during the decision interval. The most general results are those of Proposition 3: in either case, precommitted portfolio decisions under log utility become more conservative the farther in advance they are made, if risky excess returns in the farther future are mean-preserving spreads of those in the nearer future. The paper shows two dynamic returns processes under which this mean-preserving spread property holds: any ARMA$(p,q)$ process with non-zero
serial correlation, and a three-state Markov process satisfying a monotonicity condition. The sign of serial correlation is shown to be irrelevant.

Samuelson's (1991) invariance result under log utility is shown to arise due to an implausible assumption of no information availability at the decision time. The present results on decision interval effects with possible serial correlation contrast with the ambiguous results of Fischer and Pennacchi (1988), whose context differed in that they did not allow precommitted portfolio adjustments during the decision interval. Finally, the present results can be compared to those of Balvers and Mitchell (2000). They mainly assumed an ARMA(1,1) process with serial correlation and with jointly elliptically distributed innovations, allowed any concave utility function, and allowed precommitted portfolio decisions for the risky quantity to be non-constant during the decision interval; they found mixed results tending toward increased conservatism for later in the decision interval. By taking a different modelling perspective, the present paper more unambiguously finds increased conservatism for later in the decision interval, and gives an intuitive rationale for this finding.
Appendix 1

Effect of an MPS on risky asset choice of log utility (for use in proof of Proposition 1, and for proof of first inequality chain in Proposition 3)

Let the risky asset return be simply written as $b + \tilde{e}_1 + k \tilde{e}_2$ with $\tilde{e}_1$ and $\tilde{e}_2$ both zero-mean and $E(\tilde{e}_2 * e_1) = 0$, and with $k = 0$ before the spread and $k > 0$ after the spread. By Rothschild and Stiglitz (1970), any mean-preserving spread can be represented in this way. Expected utility is

$$EU(\tilde{W}) = E[\ln[R_f + w(b + \tilde{e}_1 + k \tilde{e}_2)]] \quad (A1)$$

and the first-order condition is

$$\frac{\partial EU}{\partial w} = \frac{\partial}{\partial w} E[u'(\tilde{W})][b + \tilde{e}_1 + k \tilde{e}_2] = 0. \quad (A2)$$

As is well-known, this gives $w^* > 0$ for all utility functions, given $b > 0$. For $w^*$ to decline in response to the MPS, it suffices that $dw^*/dk$, found by differentiating Equation A2, be less than or equal to zero at $k = 0$ and be less than zero for all $k > 0$. Now $dw^*/dk$ has the sign of $\frac{\partial^2 EU}{\partial w \partial k}$:

$$\frac{\partial^2 EU}{\partial w \partial k} = \frac{\partial}{\partial w} E[u''(.)][b + \tilde{e}_1 + k \tilde{e}_2] \tilde{e}_2 + u'(.) \tilde{e}_2 \quad (A3)$$

where $E_{e_1}$ refers to the expectation over the distribution of $e_1$. Here the third step uses $\tilde{W}$ $U''(.) = -U'(.)$, which is true of log utility, and the last step holds because $E(\tilde{e}_2 * e_1) = 0$. The last expression is less than zero for $k > 0$ and equal to zero for $k = 0$ since the covariance is positive (zero) for all $e_1$ because $U''(\tilde{W}) > 0$ for log utility and $\tilde{W}$ is increasing (constant) in $\tilde{e}_2$ for $k > 0$
Appendix 2

Proof of Proposition 2 and proof of second inequality chain in Proposition 3

Denote \( h_t(w_t) = E_0 \ln[R_t + w_t \bar{x}_t] \). For \( T = 2 \) the problem without a fixed-share constraint is \( \text{Max}_{w_t} h_t(w_t) \) for \( t = 1, 2 \), giving first-order conditions \( h'_t(w_t^*) = 0 \) and \( h'_2(w_2^*) = 0 \) with \( w_2^* \# w_1^* \) (\( w_2^* < w_1^* \) if \( c_1 \bar{0} 0 \)). Hence by concavity of \( h_t \), \( h'_2(w_1^*) \# 0 \) and \( h'_1(w_2^*) \$ 0 \) (again with strict inequality if \( c_1 \bar{0} 0 \)). Now the problem with a fixed-share constraint imposed is \( \text{Max}_w h_1(w) + h_2(w) \), giving first-derivative \( h_1'(w) + h_2'(w) \). This is \( \# 0 \) (\(< 0 \) if \( c_1 \bar{0} 0 \)) at \( w_1^* \) and is \( \$ 0 \) (\( > 0 \)) at \( w_2^* \), implying \( w_1^* \$ \bar{w}_2 \$ w_2^* \) (\( w_1^* > \bar{w}_2 > w_2^* \) if \( c_1 \bar{0} 0 \)).

Next, for induction assume \( \bar{w}_s \$ w_s^* \), as is true for \( s = 2 \). Then the choice of \( \bar{w}_{s+1} \) is \( \text{Max}_w h_1(w) + \ldots + h_s(w) + h_{s+1}(w) \), giving first derivative \([ h_1'(w) + \ldots + h_s'(w)] + h'_{s+1}(w) \), where the bracketed expression is zero when evaluated at \( \bar{w}_s \). Since \( \bar{w}_s \$ w_s^* \$ w_{s+1}^* \) (\( > w_{s+1}^* \) if \( c_s \bar{0} 0 \)), and \( h'_{s+1}(w_{s+1}^*) = 0 \) according to the first-order condition for choosing \( w_{s+1}^* \), by concavity \( h'_{s+1}(\bar{w}_s) \# 0 \) (\(< 0 \) if \( c_s \bar{0} 0 \)). Then the first derivative for the \( \bar{w}_{s+1} \) problem, when evaluated at \( \bar{w}_s \), is \( \# 0 \) (\(< 0 \)), so \( \bar{w}_{s+1} \# \bar{w}_s \) (\(< \bar{w}_s \)). Further, the first-derivative expression for the \( \bar{w}_{s+1} \) choice, evaluated at \( w_{s+1}^* \), is \([ h_1'(w_{s+1}^*) + \ldots + h_s'(w_{s+1}^*)] + h'_{s+1}(w_{s+1}^*) \); here the last term is zero and by Proposition 1 each term in brackets is \( \$ 0 \) (with \( h_i'(w_{s+1}^*) > 0 \) if \( c_i \bar{0} 0 \) for any of \( i = 1, \ldots, s \)) so \( \bar{w}_{s+1} \$ w_{s+1}^* \) (\( \bar{w}_{s+1} > w_{s+1}^* \) if \( c_i \bar{0} 0 \) for any of \( i = 1, \ldots, s \)).

Thus given \( \bar{w}_s \$ w_s^* \), we have found that \( \bar{w}_s \$ \bar{w}_{s+1} \$ w_{s+1}^* \), with strict inequality \( \bar{w}_s \$ w_{s+1}^* \).
$\geq \bar{w}_{s+j}$ if $c_s \bar{O} > 0$ and strict inequality $\bar{w}_{s+j} > w_{s+j}^*$ if $c_i \bar{O} < 0$ for any of $i = 1, \ldots, s$. QED.
Notes

1Lee (1990) and Gunthorpe and Levy (1994) used empirical returns data to compute mean-variance efficient mutual funds for various holding periods. Gunthorpe and Levy found that longer holding periods lead to holding lower portfolio shares in the more aggressive assets, while Lee found that the optimal share in stocks peaks at a two-year holding period.

2Balvers et al. (1990) and Cecchetti et al. (1990) show that serial correlation in stock returns, implying the absence of a random walk in stock prices, is consistent with market efficiency if the underlying fundamentals are themselves not random walks. Several papers have considered the effect of serial correlation on sequential portfolio decisions. By Mossin (1968), sequential portfolio decisions are rationally myopic even with serial correlation under log utility. Samuelson (1991), in addition to his decision interval results, showed a case with constant relative risk aversion equal to two and sequential decision making with negative serial correlation, in which a greater number of remaining decision intervals leads to a higher share currently placed in the risky asset (the conventional age effect). Balvers and Mitchell (1997) found that with sequential decision making under exponential utility, negative serial correlation gives rise to a conventional age effect while positive serial correlation gives a reverse age effect.

3For example, in the MA(1) case with $0 < \alpha < 1$, for positivity of $E_{t-1} x_i = b - \alpha u_{t-1}$ for all $u_{t-1}$ it is necessary and sufficient that this be positive when $u_{t-1}$ equals its upper bound $u^+$, and hence $u^+ <$
And for \( Pr(x_t < 0) > 0 \) always (otherwise infinite positive returns could be gotten by borrowing riskfree infinitely and investing infinitely in the risky asset), it is necessary and sufficient that this hold even when \( u_{t-1} \) equals its lower bound \( u^* \) in the expression

\[
\tilde{x}_t = b - \bar{a}u_{t-1} + \tilde{u}_t.
\]

Then it will hold if and only if \( \tilde{u}_t \) can take on a value giving

\[
b - \bar{a}u^* + \tilde{u}_t < 0,
\]

which it can if and only if

\[
b - \bar{a}u^* + u^* < 0.
\]

Thus it is required that

\[
u^* < -b / (1-\bar{a}).
\]
Bibliography


Effects of Decision Interval on Optimal Intertemporal Portfolios

With Serially Correlated Returns

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Effects of Decision Interval on Optimal Intertemporal Portfolios

With Serially Correlated Returns

Abstract

This paper considers portfolio choice when decisions are made for several future time periods all at once. The risky asset share sequence must be precommitted for the entire decision interval, either constrained (as in Samuelson (1991)) or not constrained (as in Balvers and Mitchell (2000)) to be constant across time periods within the interval. For a broad, plausible class of dynamic returns processes, contrary to Samuelson, under log utility the decisions for the more distant future are more conservative. This class is exemplified by autocorrelated ARMA(p,q) processes and finite-state Markov processes. The source of Samuelson's contrary result is elucidated.